SATELLITE ORBITS

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INTRODUCTION

These notes provide a brief historical account of the discovery of the laws governing the motion of the planets about the Sun. These laws; deduced by Johannes Kepler (1571–1630) after years of laborious calculations using planetary positions observed by Tycho Brahe (1546–1601). Isaac Newton (1642–1727) showed Kepler's Laws to be outcomes of his laws of motion and universal gravitation. These notes show how Newton's laws give an equation of motion that describes the orbits of the planets about the Sun, moons about planets, the orbits of artificial satellites of the Earth and the motion of ballistic missiles and inter-planetary flight. This equation of motion, given in the form of a second order, non-linear, vector differential equation describes the N-body problem of astrodynamics. It does not have a direct solution, but certain physical realities (e.g., relative masses of a satellite and the Earth, small orbit perturbing effects of the Sun, Moon and planets, etc.) allow simplifying assumptions when dealing with Earth-orbiting satellites. So, in practice, we deal with the two-body problem (Earth–Satellite) and its differential equation of motion – a much simpler problem to solve.

The solution of the two-body problem in these notes – formed from Newton's equations and first solved by Newton – reveals that (i) satellite orbits are elliptical, (ii) the Earth-satellite radius vector sweeps out areas at a constant rate, and (iii) the square of the period of an orbit is proportional to the cube of the mean orbital distance. Thus, these notes provide a demonstration that Kepler's Laws are an outcome of Newton's laws of motion and universal gravitation. The solution of the two-body problem in these notes relies heavily upon the use and manipulation of vectors, and as an aid to understanding a short Appendix on vectors is included.

In the derivations of the parameters of satellite orbits, these notes closely follow the text: *Fundamentals of Astrodynamics,* by Roger R. Bate, Donald D. Mueller and Jerry E. White (Dover Publications, Inc., New York, 1971), professors at the Department of Astronautics and Computer Science, United States Air Force Academy. Several students of these professors were astronauts in NASA's *Apollo* mission to the Moon.

These notes also contain definitions and explanations of coordinate systems pertinent to planetary and satellite orbital mechanics and a description of the Keplerian orbital elements. These orbital elements allow the computation of the instantaneous position of a satellite in its orbit around the Earth.

KEPLER'S PLANETARY LAWS

Johannes Kepler was born in Weil der Stadt in Württemburg (now part of Germany) in December 1571 and was a gifted young child. A scholarship, reserved for promising male children of limited means, enabled him to attend high school; later transferring to a monastic Latin school and then to the University of Tübingen where he studied under Michael Mästlin – one of the earliest advocates of the Copernican[1] system. After graduation he took up a post in mathematics and astronomy at the Protestant school in Graz, Austria where he embarked on his life-long search for a geometrical explanation of the motion of the planets in a Sun-centred Copernican system. Astronomical measurements of planetary positions were vital to Kepler's studies, and fortuitously, after being ordered to leave Graz by the Catholic archduke for his Lutheran beliefs, he was invited to continue his research in a working collaboration with the wealthy Danish astronomer Tycho Brahe at his castle in Prague, whose patron was Emperor Rudolph II. Their relationship was strained; the aristocratic Tycho possessed the most modern and accurate instruments and was an accomplished observer who had accumulated extensive planetary observations. But he lacked the mathematical skills to interpret them and treated Kepler as his assistant. Kepler, on the other hand, was a skilled mathematician but lacked data to work with.

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[1] The heliocentric system proposed in 1543 by Nicholas Copernicus (1473–1543). This Sun-centred system was in opposition to the prevailing Christian dogma of an Earth-centred universe where the apparent irregular motion of the planets was explained by a complicated set of epicycles known as Ptolemaic theory.

He was also young, newly married and poor and relying on the older Tycho for his livelihood, and he also wished to be regarded as Tycho's equal; not just an assistant. To ease Kepler's frustration, Tycho assigned him to study the orbit of Mars, which appeared to be the least circular of the observed planetary orbits. Kepler reduced Tycho's geocentric angular observations of Mars to a set of heliocentric Mars-Sun distances. As part of this reduction process, Kepler established that the path of Mars lay in a plane that passed through the Sun. Kepler, using only a straightedge and compass, then proceeded to construct possible orbital positions using the traditional mechanism of deferent, epicycle and eccentric[2]. His constructions lead to an unexpected curve – an ellipse – and in 1609 he published his results and his two laws on planetary motion in *Astronomia Nova* (New Astronomy):

- **Law 1** (the Ellipse Law) – the orbital path of a planet is an ellipse, with the Sun at a focus.

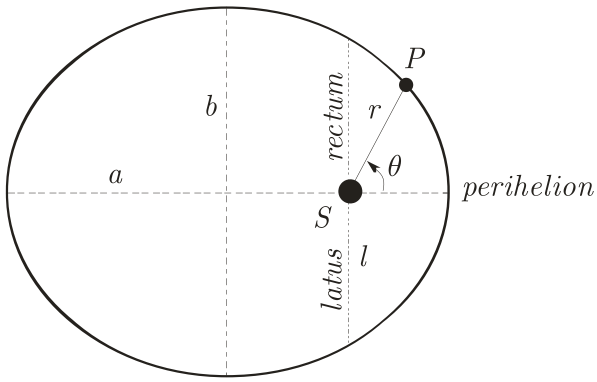

Figure 1: Kepler's first law

Figure 1 shows a planet P in its elliptical orbit around the Sun S at a focus of the ellipse whose semi-major and semi-minor axes are a and b respectively. l is the semi-latus rectum of the orbit, and perihelion is the point on the orbit when the planet is closest to the Sun. The distance r from the Sun to the planet is given by the equation

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta} = \frac{l}{1 + e\cos\theta} \tag{1}$$

[2] An eccentric is a circle (or circular orbit) whose centre is offset from the Sun; an epicycle is a circle whose centre moves around another circle known as the deferent. In Ptolemaic theory, planets moved around epicycles that moved around deferents or eccentrics.

where θ , known as the true anomaly is the angle between the radius $r = SP$ and the major axis, measured positive anticlockwise from perihelion, and e is the eccentricity of the orbital ellipse and

$$e^2 = \frac{a^2 - b^2}{a^2} \tag{2}$$

- **Law 2** (the Area Law) – the line joining the planet to the Sun sweeps out equal areas in equal times.

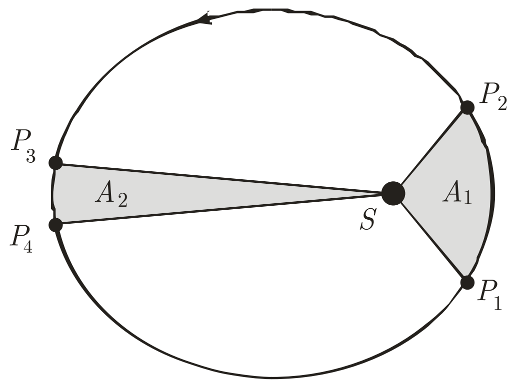

Figure 2: Kepler's second law

Figure 2 shows A_1 , the area swept out by the radius vector (the line SP) as the planet moves from P_1 to P_2 in time t_1 and the area A_2 as the planet moves from P_3 to P_4 in time t_2 . If $t_1 = t_2$ then $A_1 = A_2$ and the planet is moving faster from P_1 to P_2 than it is from P_3 to P_4 . Kepler's second law means that the sectorial area velocity is constant, or

$$\frac{dA}{dt} = \text{constant} \tag{3}$$

In polar coordinates r, θ the sector of a circle of radius r is $A = \frac{1}{2} r^2 \theta$ and a differentially small element of area $dA = \frac{1}{2} r^2 \theta \, d\theta$ thus equation (3) can be written as

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \tag{4}$$

or

$$r^2 \frac{d\theta}{dt} = C \tag{5}$$

where the constant C denotes a doubled-area.

Between 1609 and 1618, Kepler satisfied himself that the orbits of the six primary planets were ellipses with the Sun at one focus and in 1618 published further results of his work and his third law in *Harmonice Mundi* (Harmonies of the World) a series of five books

- **Law 3** (the Period Law) – the square of the period of a planet is proportional to the cube of its mean distance from the Sun.

  Kepler's third law can be expressed mathematically as

$$\frac{(\text{orbital period})^2}{(\text{semi-major axis})^3} = \frac{T^2}{a^3} = \text{ constant} \tag{6}$$

These three laws are the basis of celestial mechanics and Kepler, whilst having no idea of the forces governing the motion of the planets, proved that the planets have a certain regularity of motion and that a force is associated with the Sun. In 1627, Kepler published the *Rudolphine Tables* of planetary motion, named for his benefactor Emperor Rudolph II. These astronomical tables, based on Tycho's observations and Kepler's laws, were the most accurate yet produced and gave astronomy a new mathematical precision.

It is interesting to note that Kepler's mathematical analysis was completed without the aid of logarithms, which were not invented until 1614, by Napier (1550–1617) and that Tycho's observations had all been made with the naked eye, before the first use of the telescope in astronomy in 1610 by Galileo (1564–1642).

Whilst Kepler did not discover the force that caused planetary motion, he did discover that their motions constituted a system, and it was his Third Law that led Isaac Newton to discover the law of universal gravitation some 50 years later.

NEWTON'S LAWS

Isaac Newton was born in the English industrial town of Woolsthorpe, Lincolnshire, on Christmas Day of 1642 – the year that Galileo died. He was not expected to live long, due to his premature birth, and he later described himself as being so small at birth he could fit in a quart pot. Newton's father died before his birth and his mother remarried, placing young Isaac in the care of his grandmother. Newton as a young child was an unremarkable student, but in his teenage years he demonstrated some intellectual promise and curiosity and began preparing himself for university.

In 1661 he attended Trinity College at Cambridge University, where his uncle had been a student, and part-way through his studies in 1665, the university was closed because of the bubonic plague. Newton returned to Lincolnshire and in a period, that Newton later called his *annus mirabilis* (miraculous year), he formulated his laws of motion and gravitation. When the university reopened in 1667, Newton returned to his studies and was greatly influenced by Isaac Barrow who had been named the Lucasian[3] Professor of Mathematics. Barrow recognised Newton's extraordinary mathematical talents and when he resigned his position in 1669, he nominated Newton as his successor.

Newton's first studies as Lucasian Professor were on optics and light where he demonstrated that white light was composed of a spectrum of colours that could be seen when light was refracted by a prism. He proposed a theory of light composed of minute particles, which was a contradiction of the theories of Robert Hooke[4] (1635-1702), who contended that light travelling in waves. Hooke challenged Newton to justify his theories on light, and thus began a lifelong feud with Hooke. Newton never missed an opportunity to criticise Hooke's work and refused to publish his book *Optics* until after Hooke's death.

Early in his tenure as Lucasian Professor, Newton fell into a bitter dispute with supporters of the German mathematician Gottfried Leibniz (1646–1716) over claims of priority to the invention of calculus. The two had arrived at similar mathematical principles but Leibniz published his results first, and Newton's supporters claimed he had seen Newton's papers some years before. This bitter dispute did not end until Leibniz's death.

As an undergraduate, Newton had begun formulating theories about motion, and had set out to discover the cause of the planets' elliptical motion – a fact that Kepler had discovered 50 years before. Ironically, it was an exchange of letters with Hooke, in 1679-80, which rekindled his interest in the subject. Hooke contended that the planets were diverted from their straight line paths by some central force having an inverse square distance relationship. Hooke used these letters to Newton as the basis for a claim of priority in the discovery of the law of gravitation, but there was a great difference between a contention and a proof – a proof that Newton was to supply – and Hooke's claim was

[3] The chair of mathematics founded in 1663 with money left in the will of the reverend Henry Lucas who had been a member of Parliament for the University. The first professor was Isaac Barrow and the second was Isaac Newton. It is reserved for individuals considered the most brilliant thinkers of their time; the current Lucasian Professor is Stephen Hawking.

[4] English natural philosopher who studied light, mechanics and astronomy.

rejected. In January 1684, Christopher Wren[5] (1632–1723), Hooke and Edmund Halley[6] (1656–1742) discussed at the Royal Society, whether the elliptical shape of planetary orbits was a consequence of an inverse square law of force depending on the distance from the Sun. Halley wrote that,

> *Mr Hook said that he had it, but that he would conceale it for some time so that others, triing and failing might know how to value it, when he would make it publick.* (O'Connor & Robertson 1996)

Wren doubted Hooke's claim and offered a prize of a book to the value of forty shillings to whomever could produce a demonstration within two months. Halley afterwards recalled the meeting to Newton,

> *... and this I know to be true, that in January 84, I having from the sesquialtera proportion of Kepler, concluded that the centripetall force decreased in the proportion of the squares of the distances reciprocally, came one Wednesday to town where I met with Sr Christ. Wren and Mr Hook, and falling in discourse about it, Mr Hook affirmed that upon that principle all the laws of the celestial motions were to be demonstrated.* (Cook 1998)

In August 1684, Halley visited Newton at Cambridge,

> Newton later told de Moivre[7]:
>
> *In 1684 Dr Halley came to visit him at Cambridge, after they had been some time together the Dr asked him what he thought the Curve would be that would be described by the Planets supposing the force of attraction towards the Sun to be reciprocal to the square of their distance from it. Sr Isaac replied immediately it would be an Ellipsis, the Dr struck with joy & amazement asked him how he knew it, why saith he, I have calculated it, whereupon Dr Halley asked him for his calculation without any further delay, Sr Isaac looked among his papers but could not find it, but he promised him to renew it, & then send it him ...* (Cook 1998)

In November 1684, Halley received a nine-page article *De motu corporum in gyro* (On the motion of bodies in an orbit) in which Newton showed that an elliptical orbit could arise

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[5] Greatest of English architects. Founder of the Royal Society (president 1680–82).

[6] English mathematician, geophysicist, astronomer who discovered comets move in periodic orbits, the most famous being the one named in his honour with a period of 75½ years.

[7] Abraham de Moivre (1667–1754), a French Huguenot who settled in London in 1685 and whose name is attached to a theorem of trigonometry: $(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$. In 1733, he derived the normal probability function as an approximation to the binomial law.

from an inverse square attraction of gravity. Newton also derived Kepler's second and third laws and the trajectory of a projectile under constant gravity in a resisting medium. *De motu* did not state the law of universal gravitation or Newton's three laws of motion, but the problem he solved was crucial to the development of celestial mechanics and dynamics.

Halley realised the importance of *De motu* as soon as he received it. He visited Newton again in Cambridge, suggesting that he publish his work. By the end of 1685, Newton had expanded *De motu* into two volumes, which Halley read and annotated, and that would eventually become the *Principia*[8]. In 1686, Halley gave a presentation to the Royal Society where he reported Newton's 'incomparable *Treatise of motion* almost ready for the *Press*'. The first part arrived at the Royal Society on 28 April 1686:

> Dr Vincent presented to the Society a manuscript treatise entitled *Philosophiae naturalis principia mathematica*, and dedicated to the Society by Mr Isaac Newton, wherin he gives a mathematical demonstration of the Copernican hypothesis as proposed by Kepler, and makes out all the phaenomena of the celestial motions by the only supposition of a gravitation towards the centre of the sun decreasing as the squares of the distances therefrom reciprocally.
>
> It was ordered that a letter of thanks be written to Mr Newton; and that the printing of the book be referred to the consideration of the council; and that in the meantime the book be put into the hands of Mr Halley, to make a report thereof to the council. (Cook 1998)

Newton took about two years to write the *Principia,* from the late summer of 1684 to the middle of 1687. Halley undertook the publication of Newton's work, meeting all costs from his own resources. The *Principia* contained three books; Book I, containing Newton's three laws of motion; Book II, essentially a treatment of fluid mechanics; and Book III subtitled *System of the World* where Newton set forth the principle and law of universal gravitation and used it, together with his laws of motion, to explain the motions of the planets as well as comets, the effect of the Moon on the Earth's rotation and the ocean tides. The *Principia* was Newton's masterpiece and the fundamental work of modern science.

Newton retired from academic life in 1693 and in 1696 took up a government post as warden of the Royal Mint and oversaw the re-establishment of the English currency. He resigned his post of Lucasian Professor in 1701 and was elected by Cambridge University to Parliament; serving until 1702.

[8] *Philosophiae naturalis principia mathematica* (Mathematical Principles of Natural Philosophy).

He was elected president of the Royal Society in 1703 and re-elected every year until his death in 1727. He was knighted by Queen Mary in 1705, the first scientist to receive such an honour.

In Book 1 of the *Principia,* Newton introduces his three laws of motion:

- **First Law** – Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.
- **Second Law** – The rate of change of momentum is proportional to the force impressed and is in the same direction as that force.
- **Third Law** – To every action there is always opposed an equal reaction.

In Book III of the *Principia,* Newton formulated his law of universal gravitation, which we commonly express as:

- **The Law of Universal Gravitation** – any two bodies attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them,

$$F \propto \frac{m_1 m_2}{r_{12}^2} = \frac{G m_1 m_2}{r_{12}^2} \tag{7}$$

  where m_1, m_2 are the masses of the two bodies, r_{12} is the distance between them and G is the Newtonian constant of gravitation, whose current best known value is $G = 6.67259 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$.

THE *N-BODY* PROBLEM

A satellite orbiting the Earth has a mass m and its motion in space is affected by various forces; gravitational forces caused by mass attraction (Newton's law of universal gravitation) of the bodies Earth, Moon, Sun and the planets; forces caused by atmospheric drag for low-Earth-orbiting satellites; thrust forces caused by rocket motors; forces caused by solar radiation pressure; and other forces – often called perturbative forces, as their effect tends to move or perturb a satellite from its Keplerian orbit. The equation of motion for such a satellite would be called the equation of motion for an *N-Body* system.

The equation of motion can be expressed as a vector differential equation and it is useful to develop a vector expression for Newton's law of universal gravitation.

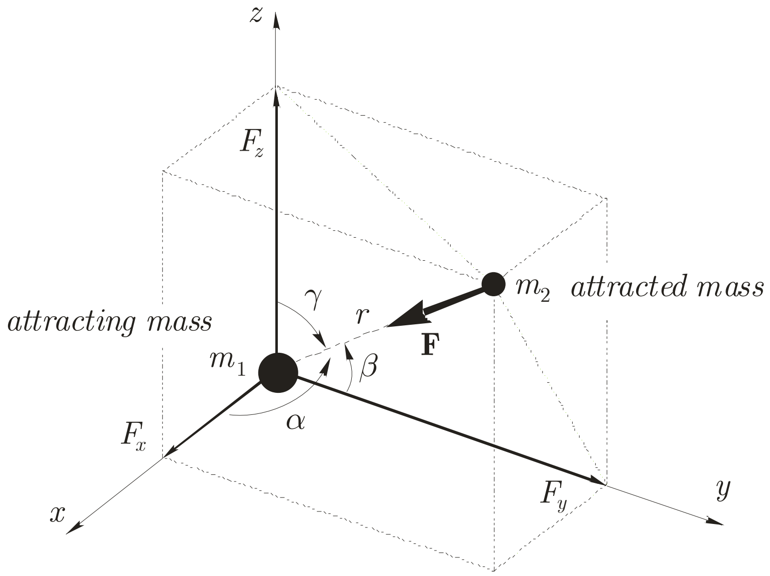

Figure 3: Gravitational force between two masses

Figure 3 shows the gravitational force $\mathbf{F}_g$ caused by mass m_1 attracting mass m_2 where the masses are a distance r apart. We may write

$$\mathbf{F}_g = -F_x\mathbf{i} - F_y\mathbf{j} - F_z\mathbf{j} \tag{8}$$

where F_x, F_y, F_z are the scalar components of $\mathbf{F}$ in the directions of the x,y,z Cartesian axes whose origin is at the centre of mass m_2. **i**,**j**,**k** are unit vectors in the direction of the Cartesian axes. Also, we may write

$$\mathbf{r}_{12} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{j} \tag{9}$$

where the distance r_{12} is the magnitude of $\mathbf{r}_{12}$ and

$$r_{12} = |\mathbf{r}_{12}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{10}$$

Also

$$\cos\alpha = \frac{x_2 - x_1}{r_{12}}; \quad \cos\beta = \frac{y_2 - y_1}{r_{12}}; \quad \cos\gamma = \frac{z_2 - z_1}{r_{12}} \tag{11}$$

Now, the scalar component F_x of $\mathbf{F}_g$ is

$$F_x = F\cos\alpha = \frac{Gm_1m_2}{r_{12}^2}\cos\alpha = \frac{Gm_1m_2}{r_{12}^2}\left(\frac{x_2 - x_1}{r_{12}}\right) = \frac{Gm_1m_2}{r_{12}^3}(x_2 - x_1) \tag{12}$$

and similarly, the components F_y and F_z are

$$F_y = \frac{Gm_1m_2}{r_{12}^3}(y_2 - y_1) \tag{13}$$

$$F_z = \frac{Gm_1m_2}{r_{12}^3}(z_2 - z_1) \tag{14}$$

Using these results in equation (8) we have the law of universal gravitation in vector notation

$$\boxed{\mathbf{F}_g = -\frac{Gm_1m_2}{r_{12}^3}\left\{(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}\right\} = -\frac{Gm_1m_2}{r_{12}^3}\mathbf{r}} \tag{15}$$

Generalizing the gravitational force

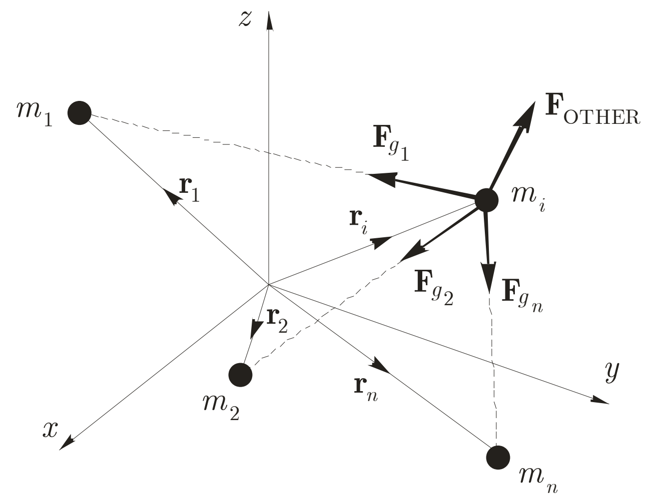

Figure 4: The *N-Body* problem

Figure 4 shows the mass bodies $m_1, m_2, \ldots, m_i, \ldots, m_n$ and the gravitational forces acting on the body of mass m_i. The resultant gravitational force acting on the body of mass m_i can be written as

$$\mathbf{F}_g = -\frac{Gm_im_1}{r_{i1}^3}\mathbf{r}_{i1} - \frac{Gm_im_2}{r_{i2}^3}\mathbf{r}_{i2} - \cdots - \frac{Gm_im_n}{r_{in}^3}\mathbf{r}_{in} = -Gm_i\sum_{\substack{j=1 \\ j\neq i}}^{n}\left\{\frac{m_j}{r_{ij}^3}\mathbf{r}_{ij}\right\} \tag{16}$$

The other external forces $\mathbf{F}_{\text{OTHER}}$ are composed of

$\mathbf{F}_{\text{DRAG}}$ (drag forces due to the atmosphere)
$\mathbf{F}_{\text{THRUST}}$ (thrust forces due to rocket motors)
$\mathbf{F}_{\text{SOLAR}}$ (solar radiation pressure)
$\mathbf{F}_{\text{PERTURB}}$ (perturbing forces due to non-spherical shape of masses)
etc.

The total force, $\mathbf{F}_{\text{TOTAL}}$ acting on the body is

$$\mathbf{F}_{\text{TOTAL}} = \mathbf{F}_g + \mathbf{F}_{\text{OTHER}} \tag{17}$$

Now, from Newton's second law of motion, where momentum = mass $\times$ velocity we may express the rate of change of momentum of m_i as

$$\frac{d}{dt}(m_i \mathbf{v}_i) = \mathbf{F}_{\text{TOTAL}}$$

where $\mathbf{v}$ is velocity; a vector quantity having magnitude v (speed) and direction.

Expanding this equation gives

$$m_i \frac{d\mathbf{v}_i}{dt} + \mathbf{v}_i \frac{dm_i}{dt} = \mathbf{F}_{\text{TOTAL}} \tag{18}$$

Note here that in equation (18) $\dfrac{dm_i}{dt}$ is the rate of change of mass, and in the case of a satellite, its mass may be changing by converting fuel into thrust. Also, velocity $\mathbf{v}$ is the rate of change of distance, i.e.,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} \tag{19}$$

and acceleration $\mathbf{a}$, also a vector quantity, is the rate of change of velocity, hence

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} \tag{20}$$

Dividing both sides of equation (18) by m_i gives

$$\frac{d\mathbf{v}_i}{dt} + \frac{\mathbf{v}_i}{m_i}\frac{dm_i}{dt} = \frac{\mathbf{F}_{\text{TOTAL}}}{m_i}$$

and re-arranging this equation gives

$$\boxed{\ddot{\mathbf{r}}_i = \frac{\mathbf{F}_{\text{TOTAL}}}{m_i} - \dot{\mathbf{r}}_i \frac{\dot{m}_i}{m_i}} \tag{21}$$

where $\dot{m}_i = \dfrac{dm_i}{dt}$ is the rate of change of mass.

Equation (21) is the second-order, non-linear, vector, <u>differential equation of motion</u>. It has no direct solution.

To simplify equation (21) and make it more amenable for solution for an Earth-orbiting satellite, we may write

$$\begin{aligned}\mathbf{F}_{\text{TOTAL}} &= \mathbf{F}_g + \mathbf{F}_{\text{OTHER}} \\ &= \mathbf{F}_g + \{\mathbf{F}_{\text{DRAG}} + \mathbf{F}_{\text{THRUST}} + \mathbf{F}_{\text{SOLAR}} + \mathbf{F}_{\text{PERTURB}} + \cdots\}\end{aligned} \tag{22}$$

and make the following assumptions

(i) the mass of the satellite, the i^{th} body, remains constant, i.e., un-powered flight hence $\dot{m}_i = 0$

(ii) $\mathbf{F}_{\text{DRAG}}$, $\mathbf{F}_{\text{THRUST}}$, $\mathbf{F}_{\text{SOLAR}}$, $\mathbf{F}_{\text{PERTURB}}$, etc., are all zero

(iii) $m_1 = M =$ mass of Earth

$m_2 = m =$ mass of satellite

$m_3 =$ mass of Moon

$m_4 =$ mass of Sun

$m_k =$ mass of planet$_k$

Hence, we are only concerned with gravitational forces and may write equation (21) as

$$\ddot{\mathbf{r}}_i = \frac{\mathbf{F}_g}{m_i} = -G\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3}\mathbf{r}_{ij} \tag{23}$$

For $i = 1$

$$\ddot{\mathbf{r}}_1 = -G\sum_{j=2}^{n} \frac{m_j}{r_{1j}^3}\mathbf{r}_{j1} \tag{24}$$

For $i = 2$

$$\ddot{\mathbf{r}}_2 = -G\sum_{\substack{j=1 \\ j\neq 2}}^{n} \frac{m_j}{r_{2j}^3}\mathbf{r}_{j2} \tag{25}$$

Now $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$, so $\ddot{\mathbf{r}}_{12} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1$ and using equations (24) and (25) gives

$$\ddot{\mathbf{r}}_{12} = -G\sum_{\substack{j=1 \\ j\neq 2}}^{n} \frac{m_j}{r_{2j}^3}\mathbf{r}_{j2} + G\sum_{j=2}^{n} \frac{m_j}{r_{1j}^3}\mathbf{r}_{j1} \tag{26}$$

Expanding equation (26)

$$\begin{aligned}\ddot{\mathbf{r}}_{12} = &-\left\{\frac{Gm_1}{r_{12}^3}\mathbf{r}_{12} + \frac{Gm_3}{r_{32}^3}\mathbf{r}_{32} + \frac{Gm_4}{r_{42}^3}\mathbf{r}_{42} + \cdots\right\} \\ &+\left\{\frac{Gm_2}{r_{21}^3}\mathbf{r}_{21} + \frac{Gm_3}{r_{31}^3}\mathbf{r}_{31} + \frac{Gm_4}{r_{41}^3}\mathbf{r}_{41} + \cdots\right\}\end{aligned}$$

and since $\mathbf{r}_{21} = -\mathbf{r}_{12}$ we have the acceleration between bodies m_1 and m_2 as

$$\ddot{\mathbf{r}}_{12} = -\frac{G\left(m_1 + m_2\right)}{r_{12}^3}\mathbf{r}_{12} - \sum_{j=3}^{n} Gm_j\left(\frac{\mathbf{r}_{j2}}{r_{j2}^3} - \frac{\mathbf{r}_{j1}}{r_{j1}^3}\right) \tag{27}$$

For the Earth $m_1 = M$ and a satellite $m_2 = m$ and $r_{12} = r =$ Earth-satellite distance, we may write equation (27) as

$$\boxed{\ddot{\mathbf{r}}_{\text{EARTH-SAT}} = -\frac{G(M+m)}{r^3}\mathbf{r} - \text{ accelerations due to Sun, Moon, planets} \qquad (28)}$$

To further simplify equation (28) it is necessary to determine the magnitude of the accelerations due to the Sun, Moon and planets compared with the acceleration between the satellite and the Earth.

Assume a satellite is in a circular orbit around the Earth at an altitude of 500 km. The acceleration, directed towards the centre of the Earth is found from equation (28) with the mass of the satellite assumed to be negligible and the acceleration due to the Sun, Moon and planets ignored

$$a = \frac{GM}{r^2} \tag{29}$$

Using $G = 6.67259 \times 10^{-11}\ \text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$, the radius of the Earth as $r = 6378000$ m and the mass of the Earth from Table 1, the acceleration is

$$a = \frac{\left(6.67259 \times 10^{-11}\right)\left(5.9742 \times 10^{24}\right)}{(6378000 + 500000)^2} = 8.427\ \text{m s}^{-2}$$

On the surface of the Earth, the acceleration, denoted by g is

$$g = \frac{\left(6.67259 \times 10^{-11}\right)\left(5.9742 \times 10^{24}\right)}{(6378000)^2} = 9.800\ \text{m s}^{-2}$$

So the relative acceleration, caused by the mass of the Earth, is $0.860g$.

Using the values given in columns 2 and 3 of Table 1 and equation (29) the relative accelerations on the satellite caused by the masses of the Sun, Moon and planets are shown in column 4.

Notice also that the effect of a non-spherical Earth (oblateness) is included for comparison.

Table 1: Comparison of relative acceleration (in g's) for a satellite orbiting the Earth at an altitude of 500 km.

	Mean distance ($\times 10^9$ m)	Mass ($\times 10^{24}$ kg)	Mass (Earth = 1)	Acceleration in g's on 500 km satellite
Sun		1989100.0	332948.0	6.1×10^{-4}
Mercury	57.9	0.3302	0.055	2.7×10^{-10}
Venus	108.2	4.8690	0.815	1.9×10^{-8}
Earth	149.6	5.9742	1.000	0.86
Mars	227.9	0.64191	0.107	7.1×10^{-10}
Jupiter	778.3	1898.8	317.833	3.3×10^{-8}
Saturn	1429.4	568.5	95.159	2.4×10^{-9}
Uranus	2875.0	86.625	14.500	7.9×10^{-11}
Neptune	4504.4	102.78	17.204	3.7×10^{-11}
Moon	0.3844 (Earth–Moon)	0.073483	0.012	3.5×10^{-6}
Earth Oblateness				1.0×10^{-3}

Notes:

1. Table 1 follows Table 1.2-1 in *Fundamentals of Astrodynamics* by Bate, Mueller and White, 1971. The values in columns 2 and 3 (mean distances and masses) are taken from the *Explanatory Supplement to the Astronomical Almanac,* edited by P. K. Seidelmann, U.S. Naval Observatory, Washington, D.C., 1992.
2. The Earth is not a spherical body with homogeneous mass density; it is actually slightly pear shaped with an equatorial bulge and variable mass density. Treating the Earth as an homogeneous spherical body will induce small errors in calculated accelerations due to gravitational attraction and this error is modelled by the Earth Oblateness value shown in column 5 of Table 1.

THE TWO-BODY PROBLEM

Equation (28) is a general expression for the relative motion of two bodies perturbed by the gravitational effects of other bodies where all other forces are ignored. It can be further simplified by the following assumptions:

(i) The bodies are spherically symmetric with homogeneous mass densities. This enables us to treat the bodies as though their masses are concentrated at their centres.

(ii) There are no external or internal forces acting on the system other than gravitational forces acting on the line joining their centres.

Hence equation (28) becomes

$$\ddot{\mathbf{r}} = -\frac{G(M+m)}{r^3}\mathbf{r} \tag{30}$$

Equation (30) is the vector differential equation of relative motion for the two-body problem, Earth (M) and satellite (m).

For $m \ll M$ we may write

$$\boxed{\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}} \tag{31}$$

Equation (31) is the two-body equation of motion that we will use in subsequent developments and is based on the assumptions above and that fact that the mass of a satellite m is very much smaller than the mass of the Earth M. We can also express the equation of motion for the two-body problem as

$$\boxed{\ddot{\mathbf{r}} + \frac{GM}{r^3}\mathbf{r} = 0} \tag{32}$$

CONSTANTS OF THE TWO-BODY MOTION

Assuming the Earth is a spherical body with homogeneous mass density, its gravitational field is spherically symmetric and a satellite moving in this conservative field possesses energy. It does not lose or gain energy but simply exchanges one form of energy, <u>kinetic</u> for another form called <u>potential energy</u>. Hence, <u>Total Energy of Motion</u> = Kinetic Energy + Potential Energy, is conserved. Also, as the satellite orbits the Earth, its radius vector sweeps out an angle and the satellite has a certain <u>angular momentum</u> (mass × angular velocity). It takes a tangential component of force to change the angular momentum of the satellite, but the only force admitted in the system is the gravitational force directed towards the centre of the Earth; so we would expect that the angular momentum of the satellite is also conserved.

Hence we have two constants of motion that can be determined from conservation of total energy, and conservation of angular momentum.

Conservation of Total Energy

The energy constant of motion can be derived as follows:

1. Using the scalar product (dot product), multiply both sides of equation (32) by $\dot{\mathbf{r}}$

$$\ddot{\mathbf{r}}\bullet\dot{\mathbf{r}} + \frac{GM}{r^3}\mathbf{r}\bullet\dot{\mathbf{r}} = 0 \qquad (33)$$

2. Now for a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then $a = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and

$$\dot{a} = \frac{da}{dt} = \tfrac{1}{2}\left(a_1^2 + a_2^2 + a_3^2\right)^{-\frac{1}{2}}\left(2a_1\frac{da_1}{dt} + 2a_2\frac{da_2}{dt} + 2a_3\frac{da_3}{dt}\right) = \frac{\mathbf{a}\bullet\dot{\mathbf{a}}}{a}$$

giving, in general, $\mathbf{a}\bullet\dot{\mathbf{a}} = a\dot{a}$. Also, $\dot{\mathbf{r}} = \mathbf{v}$ (velocity vector), $\ddot{\mathbf{r}} = \dot{\mathbf{v}}$ (acceleration vector); and we may write equation (33) as

$$\dot{\mathbf{v}}\bullet\mathbf{v} + \frac{GM}{r^3}\mathbf{r}\bullet\dot{\mathbf{r}} = 0$$

and

$$v\dot{v} + \frac{GM}{r^3}r\dot{r} = 0 \qquad (34)$$

noting that $v = |\mathbf{v}|$ and $\dot{v} = |\dot{\mathbf{v}}|$ and similarly for r and $\dot{r}$.

3. Noticing that $\frac{d}{dt}\left(\frac{v^2}{2}\right) = \frac{d}{dv}\left(\frac{v^2}{2}\right)\frac{dv}{dt} = v\dot{v}$ and $\frac{d}{dt}\left(-\frac{GM}{r}\right) = \frac{d}{dr}\left(-\frac{GM}{r}\right)\frac{dr}{dt} = \frac{GM}{r^2}\dot{r}$

equation (34) can be written as

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) + \frac{d}{dt}\left(-\frac{GM}{r}\right) = 0$$

or

$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{GM}{r}\right) = 0 \qquad (35)$$

4. If the time rate-of-change of an expression is zero, as it is in equation (35), that expression must be a constant. Call this constant the <u>Total Energy of Motion</u> denoted E_M (Bate, Mueller, White 1971)

$$\boxed{E_M = \frac{v^2}{2} - \frac{GM}{r}} \qquad (36)$$

The Total Energy of Motion E_M is the sum of the kinetic energy per unit mass $\left(\frac{v^2}{2}\right)$ and the potential energy per unit mass $\left(-\frac{GM}{r}\right)$. Note that the potential energy of a satellite will always be negative, since in the formulation of the equation, the reference point for potential energy (where the potential energy will be zero) is at infinity.

Conservation of Angular Momentum

The angular momentum constant of motion can be derived as follows:

1. Using the vector product (cross product), multiply both sides of equation (32) by $\mathbf{r}$

$$\mathbf{r}\times\ddot{\mathbf{r}}+\mathbf{r}\times\frac{GM}{r^3}\mathbf{r}=0 \tag{37}$$

2. Now, since in general $\mathbf{a}\times\mathbf{a}=0$ (from the rules of vector cross products), the second term in equation (37) is zero, giving

$$\mathbf{r}\times\ddot{\mathbf{r}}=0 \tag{38}$$

3. Noticing that $\frac{d}{dt}(\mathbf{r}\times\dot{\mathbf{r}})=\mathbf{r}\times\frac{d\dot{\mathbf{r}}}{dt}+\frac{d\mathbf{r}}{dt}\times\mathbf{r}=\mathbf{r}\times\ddot{\mathbf{r}}+\dot{\mathbf{r}}\times\mathbf{r}=\mathbf{r}\times\ddot{\mathbf{r}}$ and $\dot{\mathbf{r}}=\mathbf{v}$ then equation (38) becomes

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{v})=0 \tag{39}$$

The expression $\mathbf{r}\times\mathbf{v}$, which must be a constant of the motion, since from equation (39) its time derivative is zero, is called the Angular Momentum vector and is denoted by **h**.

$$\boxed{\mathbf{h}=\mathbf{r}\times\mathbf{v}} \tag{40}$$

Since **h** is the vector cross product of **r** and **v**, it must always be normal to the plane containing both **r** and **v**. But **h** is a constant vector, so **r** and **v** must always remain in the same plane. We can conclude, then, that the satellite's motion must be confined to a plane fixed in space and passing through the centre of mass of the Earth. This is the satellite's orbital plane.

A useful expression for the magnitude of **h** can be found by investigating the angle between the vectors **r** and **v** in the orbital plane.

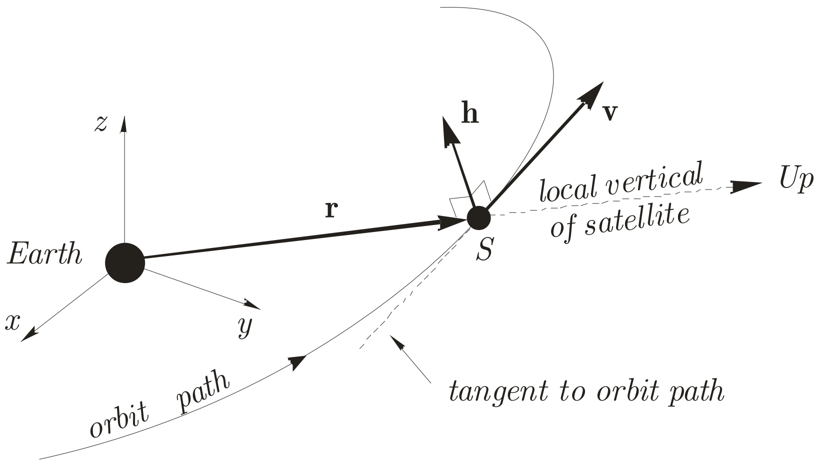

Figure 5: Orbit plane and the vectors $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{h}$

Figure 5 shows a satellite S in orbit around the Earth. The vectors $\mathbf{r}$ and $\mathbf{v}$ lie in the orbit plane and the Angular Momentum vector $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ is normal to the orbital plane. The local vertical of a satellite is in the direction of the radius vector $\mathbf{r}$ and this defines the Up direction. The local horizon plane of the satellite will be a plane normal to the local vertical and normal to the orbital plane. $\mathbf{h}$ will lie in the local horizontal plane and a local horizontal is a line in the local horizontal plane and normal to the local vertical.

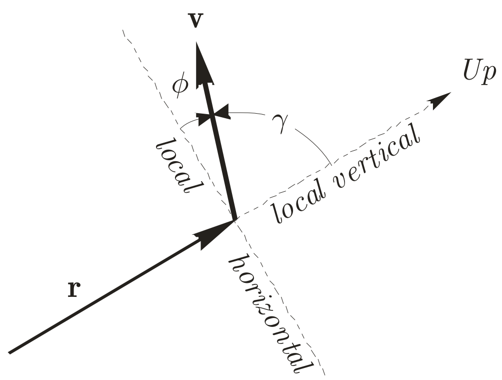

Figure 6: Flight path angle ϕ

Figure 6 shows the flight angle $\phi = 90^{\circ} - \gamma$ where γ is the angle between the radius vector $\mathbf{r}$ and the velocity vector $\mathbf{v}$. Now from the definition of a vector cross product, we may write

$$\mathbf{r} \times \mathbf{v} = rv \sin \gamma \, \hat{\mathbf{h}} = \mathbf{h}$$

where $\hat{\mathbf{h}}$ is a unit vector in the direction of **h**. Hence $rv\sin\gamma$ is the magnitude of **h**, the Angular Momentum h and

$$\boxed{h = rv\cos\phi \qquad (41)}$$

Note here that the sign of ϕ will be the same sign as $\mathbf{r}\bullet\mathbf{v}$. Figure 7 shows a satellite in an elliptical orbit around the Earth. At perigee, the flight path angle $\phi = 0^\circ$, the local vertical is normal to the velocity vector **v** and $h = rv$. At positions S_1 and S_2 the flight path angle $\phi > 0$ and at S_3 the flight path angle $\phi < 0$.

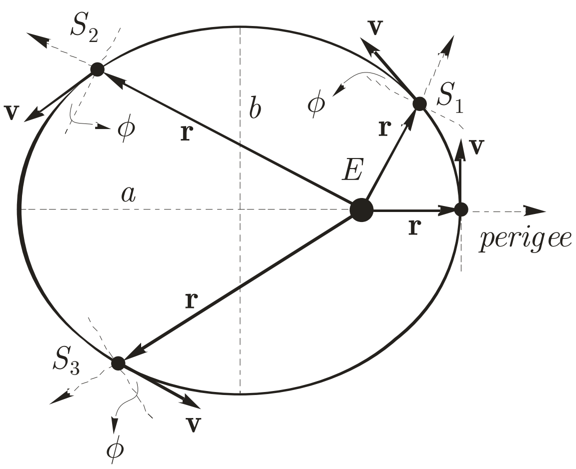

Figure 7: Flight path angles ϕ on an elliptical orbit

THE TRAJECTORY EQUATION

The solution of the equation (32) requires integration and from this we may determine the size and shape of the orbit and how the satellite moves around the orbit

Recall equation (31)

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}$$

The vector cross product $\ddot{\mathbf{r}} \times \mathbf{h}$ is

$$\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{GM}{r^3}(\mathbf{r} \times \mathbf{h}) = \frac{GM}{r^3}(\mathbf{h} \times \mathbf{r}) \qquad (42)$$

Now $\dfrac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \dot{\mathbf{r}} \times \dfrac{d\mathbf{h}}{dt} + \dfrac{d\dot{\mathbf{r}}}{dt} \times \mathbf{h}$ and since $\mathbf{h}$ must remain constant then $\dfrac{d\mathbf{h}}{dt} = 0$.
Therefore $\dfrac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \dfrac{d\dot{\mathbf{r}}}{dt} \times \mathbf{h} = \ddot{\mathbf{r}} \times \mathbf{h}$. Using this result in the left-hand-side of equation (42) gives

$$\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \frac{GM}{r^3}(\mathbf{h} \times \mathbf{r}) \tag{43}$$

Using the conservation of angular momentum [see equation (40)], the right-hand-side of equation (43) is

$$\frac{GM}{r^3}(\mathbf{h} \times \mathbf{r}) = \frac{GM}{r^3}(\mathbf{r} \times \mathbf{v}) \times \mathbf{r} \tag{44}$$

and from the rules for vector triple products and previous results

$$\begin{aligned}(\mathbf{r} \times \mathbf{v}) \times \mathbf{r} &= (\mathbf{r} \bullet \mathbf{r})\,\mathbf{v} - (\mathbf{v} \bullet \mathbf{r})\,\mathbf{r} \\ &= r^2\mathbf{v} - (\dot{\mathbf{r}} \bullet \mathbf{r})\,\mathbf{r} \\ &= r^2\mathbf{v} - (\dot{r}r)\,\mathbf{r}\end{aligned}$$

Hence, equation (44) becomes

$$\begin{aligned}\frac{GM}{r^3}(\mathbf{h} \times \mathbf{r}) &= \frac{GM}{r^3}\left\{r^2\mathbf{v} - (\dot{r}r)\,\mathbf{r}\right\} \\ &= \frac{GM}{r}\mathbf{v} - \frac{GM}{r^2}\dot{r}\,\mathbf{r} \\ &= GM\left(\frac{r\,\mathbf{v} - \dot{r}\,\mathbf{r}}{r^2}\right)\end{aligned} \tag{45}$$

Now $\dfrac{d}{dt}\left(\dfrac{\mathbf{r}}{r}\right) = \dfrac{r\dfrac{d\mathbf{r}}{dt} - \mathbf{r}\dfrac{dr}{dt}}{r^2} = \dfrac{r\,\mathbf{v} - \dot{r}\,\mathbf{r}}{r^2}$, hence equation (45) becomes

$$\frac{GM}{r^3}(\mathbf{h} \times \mathbf{r}) = GM\,\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \tag{46}$$

Substituting equation (46) into equation (43) gives

$$\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = GM\,\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

and integrating both sides of this vector equation gives

$$\dot{\mathbf{r}} \times \mathbf{h} = \frac{GM}{r}\mathbf{r} + \mathbf{B} \tag{47}$$

where $\mathbf{B}$ is a vector constant of integration.

Multiplying both sides of equation (47) by **r** (using vector scalar product) gives a scalar equation

$$\mathbf{r}\bullet(\dot{\mathbf{r}}\times\mathbf{h}) = \mathbf{r}\bullet\left(\frac{GM}{r}\mathbf{r}\right) + \mathbf{r}\bullet\mathbf{B} \tag{48}$$

Using the rules for vector triple products $\mathbf{r}\bullet(\dot{\mathbf{r}}\times\mathbf{h}) = \mathbf{h}\bullet(\mathbf{r}\times\dot{\mathbf{r}}) = \mathbf{h}\bullet(\mathbf{r}\times\mathbf{v}) = \mathbf{h}\bullet\mathbf{h} = h^2$

and equation (48) becomes

$$\mathbf{r}\bullet(\dot{\mathbf{r}}\times\mathbf{h}) = h^2 = \frac{GM}{r}\mathbf{r}\bullet\mathbf{r} + \mathbf{r}\bullet\mathbf{B} \tag{49}$$

Now, since $\mathbf{r}\bullet\mathbf{r} = r^2$ and $\mathbf{r}\bullet\mathbf{B} = |\mathbf{r}||\mathbf{B}|\cos\theta = rB\cos\theta$ where the true anomaly θ is the angle between vectors **r** and **B**, equation (49) can be written as

$$h^2 = GMr + rB\cos\theta \tag{50}$$

Re-arranging equation (50) gives the trajectory equation

$$\boxed{r = \frac{\dfrac{h^2}{GM}}{1 + \dfrac{B}{GM}\cos\theta}} \tag{51}$$

Recalling equation (1)

$$\boxed{r = \frac{a\left(1-e^2\right)}{1+e\cos\theta} = \frac{l}{1+e\cos\theta}} \tag{52}$$

we note that this is the polar equation of a conic section that is an ellipse, parabola or hyperbola according as e is less than, equal to or greater than one. Since orbits of planets and satellites of the Earth are closed curves, it follows that they must be ellipses (or circles – special cases of ellipses where $e = 0$). So we may conclude, by comparison with equation (52), that equation (51) is the polar equation of an ellipse with the origin at a focus, where r is the magnitude of the radius vector, θ is the true anomaly, l is the semi-latus rectum and e is the orbit eccentricity and

$$l = a\left(1-e^2\right) = \frac{h^2}{GM} \tag{53}$$

$$e = \frac{B}{GM} \tag{54}$$

This verifies Kepler's First Law: *the orbital path (of a planet) is an ellipse...*

B is a vector directed towards perigee, the closest point, on the elliptical orbit, to the Earth which is at the focus of the ellipse. **e**, the eccentricity vector, is also directed towards perigee, and we may write

$$\mathbf{e} = \frac{\mathbf{B}}{GM} \tag{55}$$

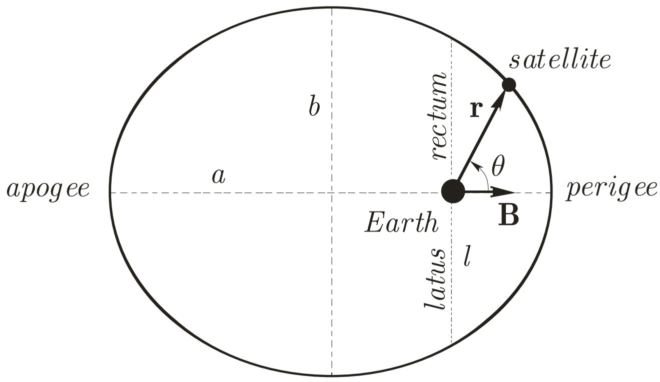

Figure 8: Orbital ellipse

Now, re-arranging equation (47) we have

$$\mathbf{B} = \dot{\mathbf{r}} \times \mathbf{h} - \frac{GM}{r}\mathbf{r} = \mathbf{v} \times \mathbf{h} - \frac{GM}{r}\mathbf{r}$$

and using equation (55) gives

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{GM} - \frac{\mathbf{r}}{r} \tag{56}$$

h can be eliminated from equation (56) using equation (40) and the rules for vector triple products

$$\begin{aligned} \mathbf{v} \times \mathbf{h} &= \mathbf{v} \times (\mathbf{r} \times \mathbf{v}) \\ &= (\mathbf{v} \bullet \mathbf{v})\,\mathbf{r} - (\mathbf{v} \bullet \mathbf{r})\,\mathbf{v} \\ &= v^2\mathbf{r} - (\mathbf{r} \bullet \mathbf{v})\,\mathbf{v} \end{aligned}$$

Substituting this result into equation (56) and re-arranging gives an expression for the eccentricity vector as

$$\mathbf{e} = \frac{1}{GM}\left[\left(v^2 - \frac{GM}{r}\right)\mathbf{r} - (\mathbf{r} \bullet \mathbf{v})\,\mathbf{v}\right] \tag{57}$$

This result is used in the determination of the elements (or parameters) of an orbit.

RELATING THE CONSTANTS OF MOTION E_M AND h TO THE GEOMETRY OF AN ORBIT

From equation (53) we see that the semi-latus rectum of the orbital ellipse is

$$l = \frac{h^2}{GM} \tag{58}$$

and the orbit parameter l depends only on the Angular Momentum h, since GM is a constant.

Now from ellipse geometry

$$l = a\left(1 - e^2\right) = a(1-e)(1+e) \tag{59}$$

So we may write

$$h^2 = GMl = GMa\left(1 - e^2\right) = GMa(1-e)(1+e) \tag{60}$$

Also, with the polar equation of the ellipse $r = \dfrac{l}{1 + e\cos\theta}$ we can see that

(i) when $\theta = 0^\circ$ the satellite is a perigee, the closest point to the Earth (see Figure 8) and

$$r = r_p = \frac{l}{1 + e\cos(90^\circ)} = \frac{a(1-e)(1+e)}{1+e} = a(1-e) \tag{61}$$

(ii) when $\theta = 180^\circ$ the satellite is a apogee, the furthest point from the Earth (see Figure 8) and

$$r = r_a = \frac{l}{1 + e\cos(180^\circ)} = \frac{a(1-e)(1+e)}{1-e} = a(1+e) \tag{62}$$

At perigee or apogee of an elliptical orbit, the velocity vector (which is always tangential to the orbit path) is normal to the axis of the orbital ellipse and the flight path angle $\phi = 0$ (see Figure 7), hence from equation (41) we have

$$h = r_p v_p = r_a v_a \tag{63}$$

where v_p, v_a are satellite velocities at perigee and apogee respectively.

Writing the Total Energy of Motion equation (36) for perigee and using equation (63) gives

$$E_M = \frac{v_p^2}{2} - \frac{GM}{r_p} = \frac{h^2}{2r_p^2} - \frac{GM}{r_p} \tag{64}$$

And using equations (60) and (61) in equation (64) gives

$$E_M = \frac{GMa(1-e)(1+e)}{2a^2(1-e)^2} - \frac{GM}{a(1-e)} = \frac{GM}{2a}\left\{\frac{1+e}{1-e} - \frac{2}{1-e}\right\}$$

which reduces to

$$\boxed{E_M = -\frac{GM}{2a} \qquad (65)}$$

We can see here that for an elliptical orbit, the Total Energy of Motion is a negative quantity and the semi-major axis a of the orbit depends only on the Total Energy of Motion.

Note: Equation (65) is valid for all conic section orbits; circle, ellipse, parabola and hyperbola. If E_M is negative the orbit is circular or elliptical, if E_M is zero the orbit is parabolic and if E_M is positive the orbit is hyperbolic. We are only concerned with elliptical or circular orbits here and a more complete treatment can be found in Bate, Mueller and White (1971).

Now, h is a function of GM, a and e [see equation (60)] and E_M is a function of GM and a [see equation (65)]; so the orbital eccentricity e is a function of E_M and h. This can be shown as follows:

From equations (58) and (59) we have $l = \frac{h^2}{GM} = a\left(1-e^2\right)$ therefore $e = \sqrt{1 - \frac{l}{a}}$.

From equation (65) $a = -\frac{GM}{2E_M}$ and substitution into the equation for e above, gives

$$\boxed{e = \sqrt{1 + \frac{2h^2 E_M}{(GM)^2}} \qquad (66)}$$

PERIOD OF AN ELLIPTICAL ORBIT

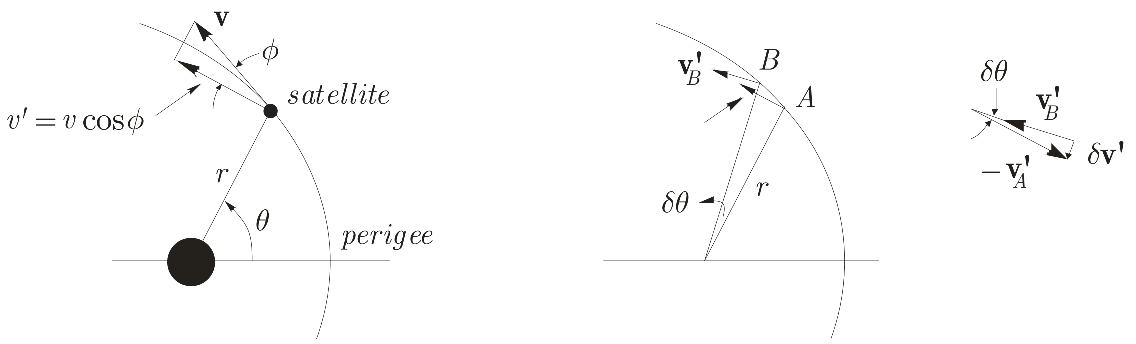

Figure 9: Horizontal component of velocity

Figure 9 shows a satellite having a velocity vector $\mathbf{v}$ with magnitude v and a horizontal component $\mathbf{v}'$ whose magnitude is $v' = v\cos\phi$ where ϕ is the flight angle. As the satellite at A moves a small distance δs along its orbit to B the radius vector sweeps out a small angle $\delta\theta$ and the horizontal component of velocity changes by a small amount $\delta\mathbf{v}' = \mathbf{v}'_B - \mathbf{v}'_A$. This change in velocity is shown as a vector triangle in Figure 9 and the magnitude of $\delta\mathbf{v}'$ can be considered as the arc of a circle of radius v'_A and $\delta v' = v'_A\,\delta\theta$. Also δs can be considered as an arc of a circle of radius r subtending an angle of $\delta\theta$ at the centre and $\delta s = r\,\delta\theta$, and the small area swept out by the radius vector is $\delta A = \frac{1}{2}r^2\,\delta\theta$.

In the limit, as B approaches A and $\delta s \to 0$ the small changes in horizontal velocity, orbit path distance and area can be written as differential relationships;

$$dv' = v'\,d\theta$$

$$ds = r\,d\theta$$

$$dA = \tfrac{1}{2}r^2\,d\theta$$

Now velocity $= \dfrac{\text{change in distance}}{\text{change in time}}$ and $v' = \dfrac{ds}{dt} = r\dfrac{d\theta}{dt}$. Using this relationship we may write the horizontal component of velocity as

$$v' = v\cos\phi = r\frac{d\theta}{dt} \tag{67}$$

Substituting equation (67) into equation (41) gives an expression for the Angular Momentum as

$$h = r^2\frac{d\theta}{dt} \tag{68}$$

that can be re-arranged as

$$dt = \frac{r^2}{h} d\theta \tag{69}$$

Now since $dA = \frac{1}{2} r^2 d\theta$ we can express equation (69) as

$$dt = \frac{2}{h} dA \tag{70}$$

This equation proves Kepler's Second Law: *the line joining the planet to the Sun sweeps out equal areas equal time,* since h is a constant for a particular orbit. We can also see the verification of Kepler's Second Law by comparing equations (68) and (5) where the constant C denoting a doubled-area is identical to h, the Angular Momentum.

During a single orbital period, denoted T, the radius vector sweeps out the area of an ellipse $A = \pi ab$ and integrating equation (70) gives the orbital period

$$T = \frac{2}{h} \int_{A=0}^{A=\pi ab} dA = \frac{2\pi ab}{h} \tag{71}$$

From ellipse geometry, the eccentricity-squared is $e^2 = \dfrac{a^2 - b^2}{a^2}$ giving $b = \sqrt{a^2\left(1 - e^2\right)}$.

But, from equation (60) we have $a^2\left(1 - e^2\right) = \dfrac{ah^2}{GM}$, hence $b = h\sqrt{\dfrac{a}{GM}}$. Substituting this result into equation (71) gives the period of an orbit as

$$\boxed{T = 2\pi \sqrt{\frac{a^3}{GM}}} \tag{72}$$

Thus the period of an elliptical orbit depends only on the size of the semi-major axis a.

Also, equation (72) is a verification of Kepler's Third Law: *the square of the period is proportional to the cube of the mean distance.* Note that a (the semi-major axis of the elliptical orbit) is the mean of r_p and r_a the lengths of the radius vectors at perigee and apogee [see equations (61) and (62)] so a can be regarded as the *mean distance* of a satellite from the Earth at the focus of the ellipse.

In the sections above we have demonstrated that the solution of the two-body problem; formulated using Newton's laws of motion and universal gravitation, yields equations defining the size and shape of elliptical orbits and their period of revolution. This is a verification of Kepler's three laws of planetary motion deduced for an analysis of relative planetary positions observed by Tycho Brahe.

The following sections show how satellite position can be computed.

TIME-OF-FLIGHT, ECCENTRIC ANOMALY, TRUE ANOMALY AND KEPLER'S EQUATION

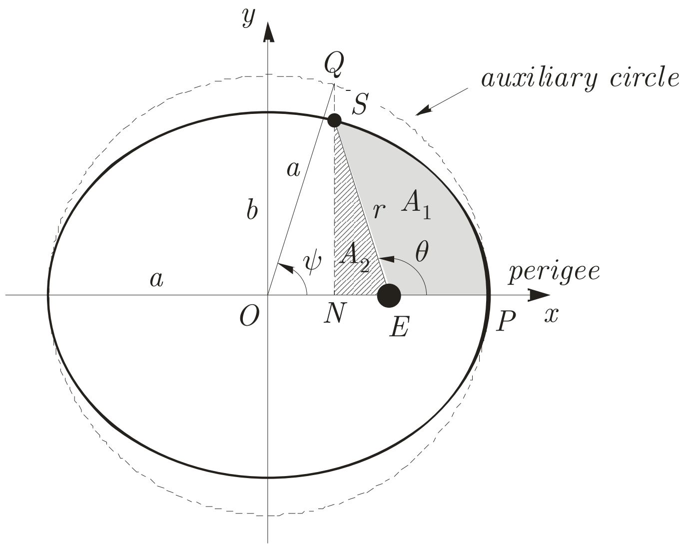

Figure 10: True anomaly θ and Eccentric anomaly ψ

Figure 10 shows a satellite S on an elliptical orbit around the Earth E. The orbital ellipse with semi-axes a and b has an auxiliary circle of radius a. The Cartesian equations of the ellipse and the auxiliary circle are

$$\text{Ellipse:}\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{Circle:}\quad x^2+y^2=a^2$$

from which

$$y_{\text{ellipse}}=\frac{b}{a}\sqrt{a^2-x^2} \quad \text{and} \quad y_{\text{circle}}=\sqrt{a^2-x^2}$$

Hence, we have the simple relationship between the y-coordinates of an ellipse and its auxiliary circle

$$\frac{y_{\text{ellipse}}}{y_{\text{circle}}}=\frac{b}{a} \tag{73}$$

In Figure 10, S and Q have the same x-coordinate, therefore by equation (73) $y_S=\dfrac{b}{a}y_Q$

Also, S is located on the ellipse by the orbital radius r and the true anomaly θ and Q on the auxiliary circle by radius a and the eccentric anomaly ψ. In orbital mechanics a satellite (on an elliptical orbit) is said to have a true anomaly and an eccentric anomaly.

The radius vector of a satellite moving in an elliptical orbit sweeps out an area πab in one revolution of period T. Let $t = t_0$ be the time that the satellite is at perigee and $t = t_k$ be the time when it is at some general point S, when the true anomaly is θ and the area swept out by the radius vector since perigee is A_1 (see Figure 10). Then – because area is swept out at a constant rate (Kepler's Second Law) – we can write

$$\frac{t_k - t_0}{A_1} = \frac{T}{\pi ab} \tag{74}$$

where $t_k - t_0$ is the time-of-flight; the time since the satellite passed through perigee at $t = t_0$.

From Figure 10, the area swept out by the radius vector is Area NSP minus Area NSE, or

$$A_1 = \text{ Area } NSP - A_2 \tag{75}$$

Now from ellipse geometry $OE = ae$, where e is the orbital eccentricity, and A_2 is a triangle with base $OE - ON = ae - a\cos\psi = a(e - \cos\psi)$ and altitude $\frac{b}{a}(a\sin\psi) = b\sin\psi$ and

$$A_2 = \frac{ab}{2}(e\sin\psi - \sin\psi\cos\psi) \tag{76}$$

Area NSP is the area under the ellipse and Area NQP is the area under the auxiliary circle, so from equation (73)

$$\text{Area } NSP = \frac{b}{a}(\text{Area } NQP) \tag{77}$$

Area NQP is the area of the sector $OQP = \tfrac{1}{2}a^2\psi$ minus Area $OQN = \tfrac{1}{2}a^2\sin\psi\cos\psi$ and substituting these results into equation (77) gives

$$\text{Area } NSP = \frac{ab}{2}(\psi - \sin\psi\cos\psi) \tag{78}$$

Now substituting equations (78) and (76) into equation (75) gives

$$A_1 = \frac{ab}{2}(\psi - e\sin\psi) \tag{79}$$

Finally, substituting equation (79) into equation (74) with the aid of equation (72) gives

$$t_k - t_0 = \sqrt{\frac{a^3}{GM}}(\psi - e\sin\psi) \tag{80}$$

Kepler introduced the term <u>mean anomaly</u> M and defined it as

$$\boxed{M \equiv \psi - e\sin\psi} \tag{81}$$

Defining the <u>mean motion</u> n as

$$\boxed{n \equiv \sqrt{\frac{GM}{a^3}}} \tag{82}$$

gives the orbital period T as

$$\boxed{T = \frac{2\pi}{n}} \tag{83}$$

and the mean anomaly M as

$$\boxed{M = n\left(t_k - t_0\right) = \psi - e\sin\psi} \tag{84}$$

Equation (84) is known as *Kepler's equation*.

The mean anomaly M is the angle in the orbital plane, with respect to the centre of a mean circular orbit having the same period T as the elliptical orbit, measured from perigee to the satellite position.

The true anomaly θ and the eccentric anomaly ψ are connected by equations that can be developed using Figure 10 as follows.

$$\cos\psi = \frac{ON}{OQ} = \frac{ae + r\cos\theta}{a} \tag{85}$$

and substituting r from equation (52) into equation (85) and simplifying gives

$$\cos\psi = \frac{e + \cos\theta}{1 + e\cos\theta} \tag{86}$$

Similarly

$$\sin\psi = \frac{NQ}{OQ} = \frac{\frac{a}{b}(r\sin\theta)}{a} = \frac{r\sin\theta}{a\sqrt{1-e^2}}$$

which reduces to

$$\sin\psi = \frac{\sqrt{1-e^2}\,\sin\theta}{1+e\cos\theta} \tag{87}$$

Dividing equation (87) by equation (86) gives

$$\boxed{\tan\psi = \frac{\sqrt{1-e^2}\,\sin\theta}{\cos\theta + e}} \tag{88}$$

Again, from Figure 10 we may write two relationships

$$r\sin\theta = \left(\frac{b}{a}\right)a\sin\psi = a\sqrt{1-e^2}\,\sin\psi$$

$$r\cos\theta = a\cos\psi - ae = a(\cos\psi - e)$$

and dividing one by the other leads to

$$\boxed{\tan\theta = \frac{\sqrt{1-e^2}\,\sin\psi}{\cos\psi - e}} \tag{89}$$

Note that the units of the angles M, θ and ψ are radians, the units of the period T are seconds and the units of the mean motion n are radians-per-second.

COORDINATE SYSTEMS

A reference system is a conceptual definition of an "ideal" Cartesian coordinate system based on some abstract principles. A conventional reference system is one where the model used to define coordinates is given in detail, e.g., a coordinate origin is defined, primary planes of reference defined, positive directions of axes defined and reference surfaces defined. Assigning coordinate values to points constitutes a realization of a reference system and having done this we now have a reference frame. Subsequent realizations (i.e., new values of coordinates) are new reference frames which may be distinguished from each other by a date or epoch. An important part of a reference frame is the mathematical definition of the method of connection between different reference frames allowing the transformation of coordinates from different measurement epochs.

A useful reference frame is one where the origin is stationary and the axes of the reference frame are motionless. Such reference frames are called inertial[9]. Some reference frames in a region of space are very close to inertial, i.e., their axes may be rotating very slowly and are sometimes called quasi-inertial.

The Heliocentric–Ecliptic Coordinate System

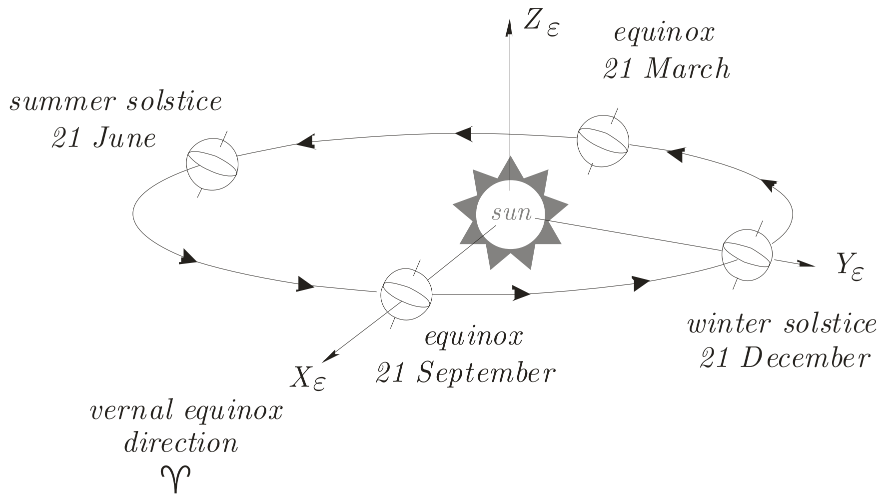

Figure 11: Heliocentric–ecliptic coordinate system

The orbital path of the Earth around the Sun is an ellipse (with the Sun at the primary focus) and the Earth–Sun orbital plane is known as the ecliptic. The Earth is rotating about its polar axis as it moves along its orbital path and the Earth's equator and its equatorial plane are inclined with respect to the ecliptic plane, i.e., the Earth's axis is not normal to the ecliptic. This angle of inclination, known as the obliquity of the ecliptic, is approximately 23½°. The intersection of the ecliptic and equatorial planes will be a line in the ecliptic plane and at two places on the Earth's orbital path, this line will pass through the centre of mass of the Sun, thus defining the direction of the equinoxes that are the two positions of the Earth on its orbit around the Sun (on or about the 21st of March and 21st

[9] In classical mechanics, an inertial reference frame is one in which Newton's First and Second Laws of Motion are valid. Newton's laws are valid in any reference frame (in space containing all the matter of the universe) that is neither rotating nor accelerating. Einstein's theory of special relativity defines inertial frames in a space-time continuum in the absence of gravitational fields rather than in absolute space, but there are finite regions in space where special relatively holds with remarkable accuracy and a reference frame in this region may be termed quasi-inertial.

September) where there will be equal periods of daylight and darkness as the Earth completes a single revolution about its axis.

The $X_\varepsilon, Y_\varepsilon, Z_\varepsilon$ reference frame of the Heliocentric–Ecliptic system has the ecliptic plane as the X_ε-Y_ε plane and the positive direction of the X_ε axis points along the line of the equinoxes, from the Sun to the Earth's position in its orbit on or about the 21st September. This direction is known as the direction of the vernal equinox and is given the astrological symbol ♈ – depicting the head and horns of a ram for the zodiac constellation Aries. ♈ is also known as the First Point of Aries.

An observer on Earth, on or about 21st March, would see the Sun rise due East and set due West, and there would be equal periods of daylight and darkness. The apparent motion of the Sun on this day would be as though it were a fixed object on the equator of a rotating celestial sphere[10] and as darkness arrives, star constellations (again considered as fixed objects on the celestial sphere) will appear to rise due East, in the same direction of sunrise. Thousands of years ago, astronomers observed the constellation Aries rising due east on this day, which also coincided with spring, and called this day the vernal equinox (*ver* is Latin for spring; *equus* and *nox* are Latin for equal and night). The direction of the vernal equinox (the direction to the First Point of Aries) to an observer on Earth is a fixed point in space at the intersection of the ecliptic and equatorial planes, having a daily motion akin to an imaginary star on the celestial sphere. Unfortunately, the First Point of Aries (and the line of the equinoxes) is not exactly fixed in space, but instead is slowly moving; and this slow motion in space is known as precession. Due to the gravitational effects of the planets on the Earth's orbit, the ecliptic is slowly moving in space and this motion causes a contribution to precession known as planetary precession. Also, the Moon is rotating about the Earth and the Earth-Moon system, with its own slowly moving centre of mss, is rotating about the Sun. This force-couple causes the Earth's axis to slowly rotate in space; much like a spinning top whose spin-axis starts to wobble as it slows. This slow rotation of the Earth's axis in space, known as luni-solar precession has a period of approximately 26,000 years and consequently, the equatorial plane has a slow periodic motion.

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[10] A sphere of infinite radius, with the Earth at its centre, and whose poles (the north and south celestial poles) are extensions of the Earth's poles and whose equator (the celestial equator) is the extension of the Earth's equator. Apparent daily motions of the Sun, Moon, planets and stars are as though they are fixed objects on the celestial sphere which is rotating about it polar axis with the same rotation rate as the Earth.

The combined effects of planetary and luni-solar precession, known as general precession, cause the line of equinoxes to slowly rotate, thus causing an apparent motion of the First Point of Aries (the direction of the vernal equinox) around the celestial equator of approximately 51" per year. This precession of the First Point of Aries was first observed by the Greek astronomer Hipparchus (190–120 BC) as an apparent annual motion of the rising and setting positions of certain stars. Due to precession, the First Point of Aries now rises in the constellation of Pisces and will move into the constellation of Aquarius in the future.

The Heliocentric-Ecliptic coordinate system is not a true inertial reference frame since precession will cause the line of equinoxes to slowly rotate in space, thus the (quasi-inertial) $X_\varepsilon, Y_\varepsilon, Z_\varepsilon$ coordinates are based on a particular year or epoch.

Geocentric–Equatorial Reference System (Conventional Celestial Reference System)

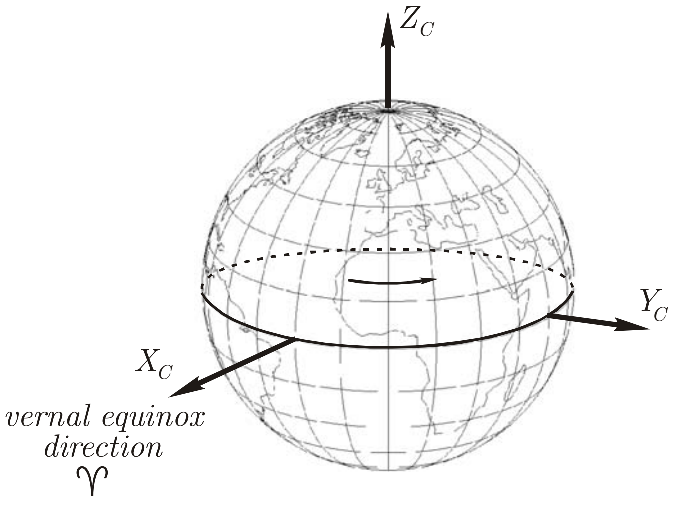

Figure 12: Geocentric–Equatorial coordinate system

The reference frame of the X_C, Y_C, Z_C Geocentric–Equatorial system has its origin at the Earth's centre of mass. The X_C - Y_C plane is the Earth's equatorial plane and the positive X_C -axis points in the vernal equinox direction (in the direction of The First Point of Aries). The positive Z_C axis points in the direction of the north pole. The Geocentric–Equatorial reference frame, also known as the Conventional Celestial Reference Frame is sometimes termed quasi-inertial since the coordinate axes are slowly rotating in space due to the effects of precession.

It is important to note here that the X_C, Y_C, Z_C reference frame is not fixed to the Earth and rotating with it. Rather, it is non-rotating with respect to the stars (except for precession of the equinoxes) and the Earth rotates relative to the reference frame.

The Right Ascension–Declination System

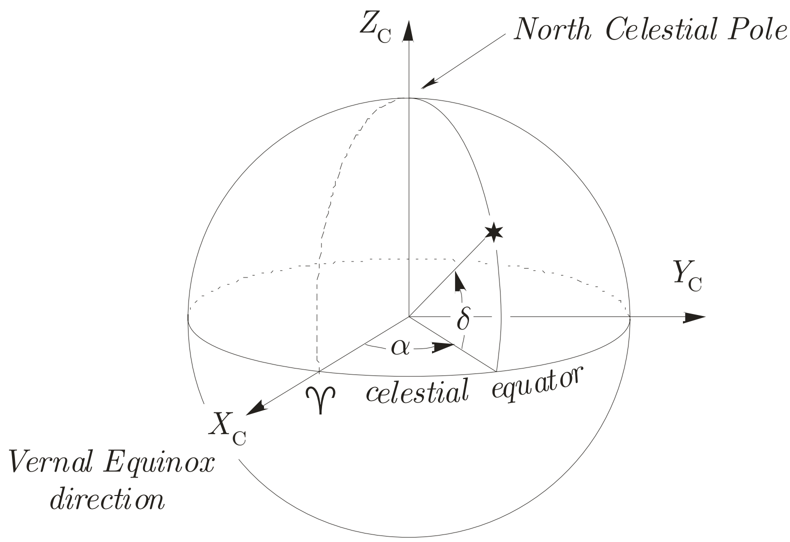

Figure 13: Right Ascension–Declination coordinate system

The quasi-inertial reference frame of the Right Ascension–Declination system is closely related to the Geocentric–Equatorial reference frame. The fundamental reference plane is the celestial equator which is the extension of the Earth's equator onto the celestial sphere. The position of an object projected onto the celestial sphere is fixed by two angles; Right Ascension denoted by α measured in the plane of the celestial equator, positive eastwards from 0° to 360° (or 0h to 24h) from the First Point of Aries; and Declination denoted by δ and measured in a plane perpendicular to the celestial equator from 0° to $\pm 90^\circ$ north or south. The origin of the Right Ascension–Declination system (i.e., the centre of the celestial sphere) may be at the Earth's centre of mass, at a point on the Earth's surface, or anywhere else. For all intents and purposes any point may be considered as the centre of a sphere of infinite radius. Astronomers use the Right Ascension–Declination reference frames (defined at particular epochs identifying the direction of the line of equinoxes) to record positions of stars. Because of the huge distances to the stars, their coordinates remain essentially unchanged even when viewed from opposite sides of the Earth's orbit around the Sun. Only measurements to a few of the closest stars (at six-monthly intervals) reveal a difference that could be attributed to parallax.

Conventional Terrestrial Coordinate System

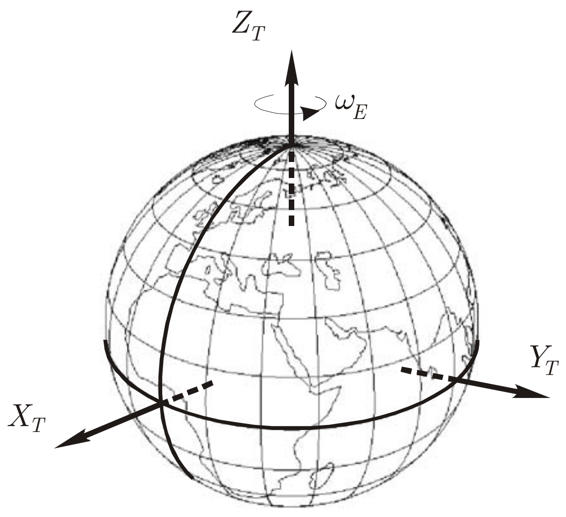

Figure 14: Conventional Terrestrial coordinate system

The reference frame of the X_T, Y_T, Z_T Conventional Terrestrial system has its origin at the Earth's centre of mass. The positive Z_C-axis points in the direction of the north pole and the Z_C-axis is coincident with the Earth's rotational axis. The X_T-Y_T plane is the Earth's equatorial plane and the positive X_T-axis points through the intersection of the Greenwich meridian and the equator. The positive Y_T-axis, advanced 90° along the equator completes a right-handed coordinate system. The X_T, Y_T, Z_T reference frame – also known as an Earth-Centred–Earth-Fixed (ECEF) reference frame – is rotating about the Z_C-axis with an angular velocity $\omega_E = 7.2921151467\text{e-}05$ radians/sec[11], i.e., the axes are fixed to the Earth and rotating with respect to the stars. The Geocentric–Equatorial system, the Right-Ascension–Declination system and the Conventional Terrestrial system all have a common primary reference plane – the Earth's equatorial plane. And the Z-axes of the three systems are coincident.

Sidereal time and solar time

The connection between the X_T, Y_T, Z_T Conventional <u>Terrestrial</u> reference frame and the X_C, Y_C, Z_C Conventional <u>Celestial</u> reference frame (Geocentric–Equatorial) is via the motion of ♈ (the First Point of Aries); which we may imagine as the regular diurnal[12]

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[11] This is the current best known value given in the World Geodetic System 1984 (WGS84) which is the reference system of the Global Positioning System (GPS).

[12] Daily motion.

motion of a star having both a Right Ascension and Declination of zero. In this sense ♈ acts as a sidereal[13] timekeeper where the imaginary meridian (or hour circle) passing through ♈ (see Figure 13) sweeps out sidereal time (hour angles) at the celestial pole as ♈ rotates with the celestial sphere. One apparent revolution of ♈ (i.e., two successive transits by ♈ of an observer's meridian), equalling one sidereal day.

Another, more familiar timekeeper is the Sun, whose daily motion regulates many of our activities. Because of the obliquity of the ecliptic and the Earth's elliptical orbit, the Sun's motion is irregular and it is not a suitable timekeeper, instead a fictitious Mean Sun is created which moves at a constant velocity around the celestial equator, with an annual period exactly equal to the period of the Earth's orbit of the Sun. The Mean Sun can be considered as an object on the celestial sphere having a constant Declination equal to zero and a Right Ascension changing at a regular rate. The Mean Sun has a regular diurnal motion and it will rise and set due East and West as the celestial sphere rotates. Successive transits of the Mean Sun of an observer's meridian define the length of a mean solar day, although, for practical timekeeping, we regard the solar day as beginning and ending as the Sun (Mean or apparent) transits an observer's lower meridian (lower transit); thus there is no date change (a count of days) during daylight hours.

The length of the sidereal day is not the same length as a solar day (or a mean solar day) and this fact can be established from the following diagram

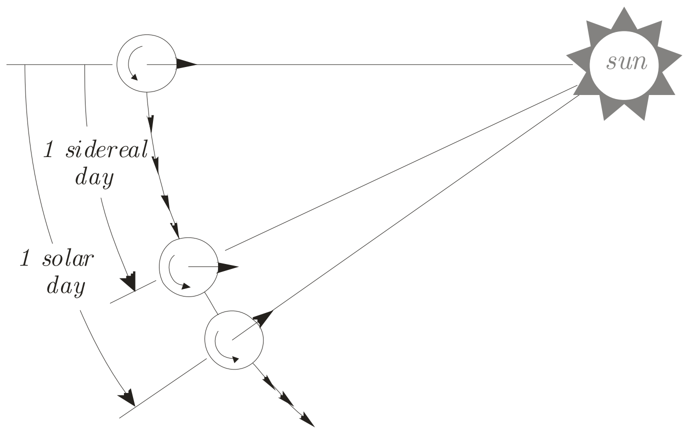

Figure 15: Solar and sidereal day

[13] From the Latin *siderius* (*sidus sideris*) meaning star. A sidereal day is the period between successive upper transits (of an observer's meridian) of the First Point of Aries

In Figure 15, at a certain instant of time, the Earth, Sun and a star (infinitely far away) all appear to be transiting an observer's meridian (the line and black arrow). The Earth is rotating about its axis as it is moving in its orbit about the Sun and one sidereal day later, the distant star will again transit the observer's meridian. A short time later the Sun will transit the observer's meridian completing one solar day. Thus the solar day is longer than the sidereal day.

In the course of one tropical year (the time between successive passages of the Earth through the vernal equinox) the Earth completes exactly one more revolution about its axis with respect to the direction of the vernal equinox than it does with respect to the Sun. Thus (using 1990 values) the tropical year has 365.2421897 mean solar days = 366.2421897 sidereal days giving the relationship between mean solar time intervals and sidereal time intervals as:

$$\text{mean solar time interval} = 1.00273790935 \times \text{sidereal time interval}$$

Using this relationship, 1 sidereal day = 23h 56m 04.09s of mean solar time, which we see as the stars rising in the sky approximately 4 minutes earlier each day.

Time diagram

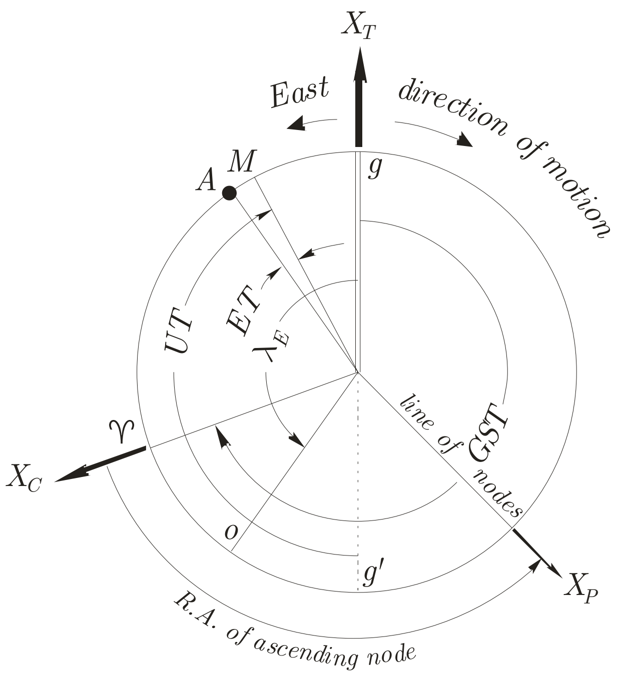

Figure 16: Time diagram

The Mean Sun (M) and the First Point of Aries ♈ are sidereal and mean time timekeepers respectively and their daily motions and the relationship between time and longitude are best explained with the aid of a time diagram. In Figure 16, the time diagram is a schematic view of the celestial sphere from a point above the north celestial pole, which is at the centre of a circle representing the celestial equator. Radial lines are either meridians of longitude if they are connected to points on the Earth's surface (*g* for Greenwich, *o* for observer), or hour circles if they are connected to objects rotating with the celestial sphere (*M* for Mean Sun, *A* for apparent Sun, ♈ for First Point of Aries). As an object (*M, A* or ♈) moves around the celestial equator, its hour circle sweeps out hour angles at the pole which are measured positive clockwise (westwards) from particular meridians and

$$1 \text{ revolution} = 360^{\circ} = 24 \text{ hours}$$

An object (*M, A* or ♈) is at upper transit when its hour angle is zero and lower transit when its hour angle is 12 hours.

Greenwich Sidereal Time (GST) is the Greenwich Hour Angle of the First Point of Aries (GHA_{Υ}). Universal Time (UT) is the Greenwich Hour Angle of the Mean Sun − 12 hours. The small difference between the Mean Sun and Apparent (or true) Sun is the Equation of Time (ET) which varies between ±(14½ to 16½ minutes) due to the elliptical orbit of the Earth and the obliquity of the ecliptic.

Right Ascension (R.A., measured positive eastwards from the direction of the First Point of Aries) and Longitude (λ, measured positive eastwards from the Greenwich meridian) are also conveniently represented on time diagrams. With the aid of Figure 16, the Longitude of the Ascending Node denoted by λ_N is

$$\lambda_N = \left(24^h - \text{GST}\right) \times 15 + \Omega = \lambda_A + \Omega \tag{90}$$

where $\lambda_A = \left(24^h - \text{GST}\right) \times 15$ is in degrees and is the longitude of the First Point of Aries, Ω (the right ascension of the ascending node) is in degrees and GST is in hours.

The transformation between coordinates in the Conventional Celestial and Conventional Terrestrial reference frames is achieved using rotation matrices, i.e.,

$$\begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} = \mathbf{R}_{\lambda_A} \begin{bmatrix} X_T \\ Y_T \\ Z_T \end{bmatrix} \quad \text{where} \quad \mathbf{R}_{\lambda_A} = \begin{bmatrix} \cos\lambda_A & \sin\lambda_A & 0 \\ -\sin\lambda_A & \cos\lambda_A & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{91}$$

and

$$\begin{bmatrix} X_T \\ Y_T \\ Z_T \end{bmatrix} = \mathbf{R}_{\lambda_A}^{-1} \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} = \mathbf{R}_{\lambda_A}^{T} \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} \quad \text{where} \quad \mathbf{R}_{\lambda_A}^{T} = \begin{bmatrix} \cos\lambda_A & -\sin\lambda_A & 0 \\ \sin\lambda_A & \cos\lambda_A & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{92}$$

Noting here that the rotation matrix $\mathbf{R}_{\lambda_A}$ is orthogonal, hence its inverse is equal to its transpose.

The general relationships between Universal Time (UT), Greenwich Sidereal Time (GST), Right Ascension and Longitude established above from the time diagram (Figure 16) need some qualification.

Firstly, the First Point of Aries (the equinox) is not fixed on the celestial sphere, but instead has a slow westward motion due to general precession. Thus there are two sidereal times; Greenwich Means Sidereal Time (GMST) and Greenwich Apparent Sidereal Time (GAST) and these two differ by a small amount known as the <u>equation of the equinoxes</u>. GAST is the Greenwich hour angle of the apparent or true equinox, and can be determined at observatories by observation to fixed stars. GMST is the Greenwich hour angle of a mean equinox (the mean equinox of date).

Secondly, Universal Time (UT) conforms closely to the mean diurnal motion of the Sun, but not exactly, and it is linked to GMST by a defined relationship. Hence UT can be determined by measurements to the stars. The uncorrected observed rotational timescale dependent upon the place of observation is designated UT0 and correcting this timescale for the effects of polar motion on the longitude of the place of observation produces UT1. UT1 is linked by formula to GMST but varies slightly due to the variable rotation rate of the Earth.

Thirdly, since January 1st, 1972 all time services have used Coordinated Universal Time (UTC), which differs from International Atomic Time[14] (TAI) by an integral number of seconds. UTC is maintained within 0.90 second of UT1 by the introduction of one-second steps (leap seconds) when necessary, usually at the end of June or December.

[14] International Atomic Time (*Temps Atomique International*, or TAI) is based on an ensemble of atomic clocks at observatories around the world. The fundamental unit of atomic time is the Système International (SI) second defined as the duration of 9,192,631,770 periods of the radiation corresponding to the transition between two hyper-fine levels of the ground state of the cesium-133 atom.

Lastly, the line of the nodes, and consequently the Right Ascension of the ascending node, is not fixed. Its location slowly varies due to the precession of the satellites orbital plane caused by the equatorial bulge of the Earth.

A more complete treatment of time and related coordinate systems can be found in Seidelmann (1992).

Perifocal Coordinate System

A convenient reference frame for describing the motion of a satellite is the $X_\omega, Y_\omega, Z_\omega$ Perifocal[15] coordinate reference frame. The origin of the coordinate system is at the Earth's centre of mass and the X_ω-Y_ω plane is the satellite's orbital plane. The positive X_ω-axis passes through perigee and the Y_ω-axis is advanced 90° in the direction of orbital motion. The Z_ω-axis completes the right-handed system.

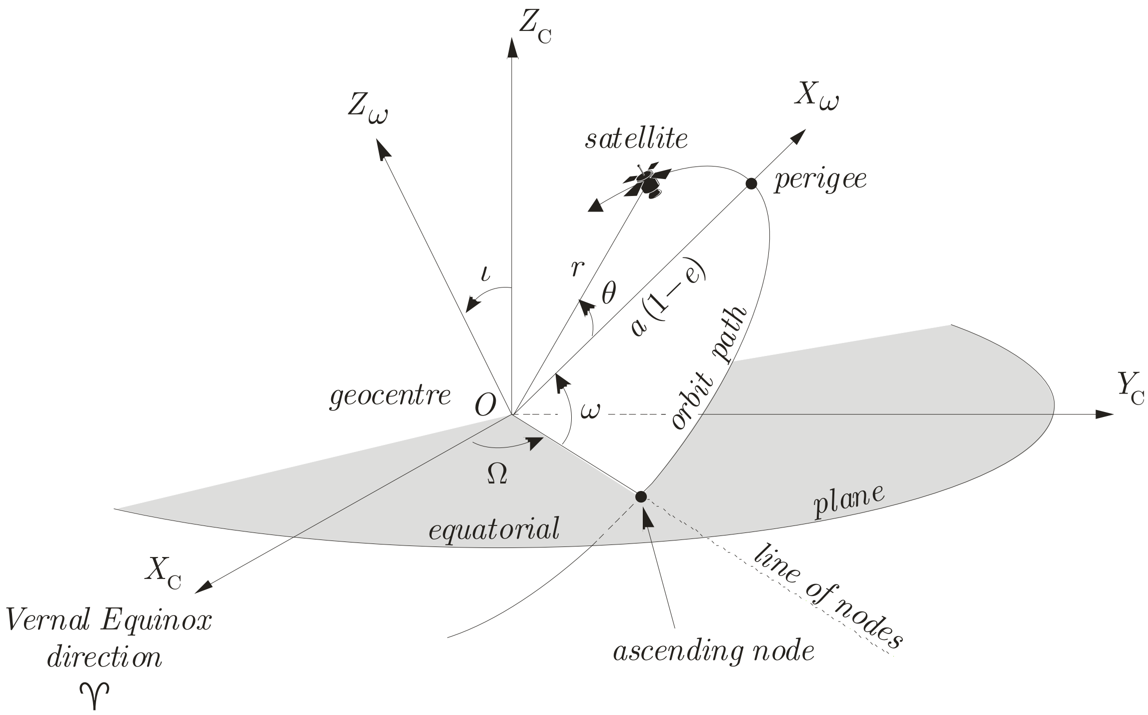

Figure 17: Perifocal Coordinate System and Satellite orbit elements

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[15] In orbital mechanics, the major axis of the orbital ellipse is known as the *line of apsides.* The nearest and furthest points on the orbit from the focus are *periapsis* and *apoapsis* respectively and known collectively as apse points. If the Earth is at the focus, the apse points are called *perigee* and *apogee* and if the Sun is at the focus then they are *perihelion* and *aphelion.* A coordinate system with the origin at the focus and the X-axis in the direction of periapsis is a *perifocal* coordinate system.

KEPLERIAN ORBITAL ELEMENTS

Given a time t_k the X_C, Y_C, Z_C Geocentric–Equatorial system coordinates of a satellite in an elliptical orbit can be computed with the aid of six independent quantities called Keplerian orbital elements (or orbital parameters). These orbital elements completely describe the size, shape and orientation of an orbit and the time t_0 of perigee passage. This classical set of orbit elements are defined with the aid of Figure 17 as follows.

1. *a,* semi-major axis – a constant defining the size of the ellipse.
2. *e,* eccentricity – a constant defining the shape of the ellipse.
3. ι , inclination – then angle between the orbital plane and the equatorial plane.
4. Ω , longitude of the ascending node[16] – the angle measured positive eastwards in the equatorial plane from the direction of the vernal equinox. This angle is also known as the Right Ascension of the ascending node.
5. ω , argument of perigee – the angle measured in the orbital plane between the line to the ascending node and the line through perigee (the X_ω-axis). The positive direction of ω is the direction of the satellite's motion.
6. t_0 , time of perigee passage – the time when the satellite was at perigee.

X_C, Y_C, Z_C GEOCENTRIC–EQUATORIAL COORDINATES AT TIME t_k USING KEPLERIAN ORBITAL ELEMENTS

Given *GM* (the product of the Newtonian constant of gravitation *G* and the mass of the Earth *M*) the computation of coordinates is achieved by firstly computing the Perifocal coordinates $X_\omega, Y_\omega, Z_\omega = 0$ at time t_k and then transforming them to the X_C, Y_C, Z_C reference frame by orthogonal rotation. The computational steps are:

1. Compute the orbital mean motion constant $n = \sqrt{\dfrac{GM}{a^3}}$
2. Compute the mean anomaly $M = n\left(t_k - t_0\right)$
3. Solve Kepler's equation $M = \psi - e\sin\psi$ for the eccentric anomaly ψ

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[16] The intersection of the orbital plane and the equatorial plane is *the line of nodes*. The ascending node is the point where the satellite passes from below to above the equatorial plane. The descending node is the point where the satellite passes from above to below the equatorial plane.

4. Compute the true anomaly θ from $\tan\theta = \dfrac{\sqrt{1-e^2}\sin\psi}{\cos\psi - e}$

5. Compute radius vector $r = \dfrac{a\left(1-e^2\right)}{1+e\cos\theta}$

6. Compute Perifocal coordinates $X_\omega, Y_\omega, Z_\omega = 0$ from $X_\omega = r\cos\theta$

$$Y_\omega = r\sin\theta$$

7. Transform $X_\omega, Y_\omega, Z_\omega = 0$ Perifocal coordinates to X_C, Y_C, Z_C Geocentric–equatorial coordinates using $\mathbf{x}_C = \mathbf{R}\mathbf{x}_\omega$ or

$$\begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} = \begin{bmatrix} \begin{array}{c}\cos\omega\cos\Omega \\ -\sin\omega\cos\iota\sin\Omega\end{array} & \begin{array}{c}-\sin\omega\cos\Omega \\ -\cos\omega\cos\iota\sin\Omega\end{array} & \sin\iota\sin\Omega \\ \begin{array}{c}\cos\omega\sin\Omega \\ +\sin\omega\cos\iota\cos\Omega\end{array} & \begin{array}{c}-\sin\omega\sin\Omega \\ +\cos\omega\cos\iota\cos\Omega\end{array} & -\sin\iota\cos\Omega \\ \sin\omega\sin\iota & \cos\omega\sin\iota & \cos\iota \end{bmatrix} \begin{bmatrix} X_\omega \\ Y_\omega \\ Z_\omega \end{bmatrix} \tag{93}$$

Note that the rotation matrix $\mathbf{R}$ has been derived in the following manner. Referring to Figure 17, the X_C, Y_C, Z_C axes can be rotated into the $X_\omega, Y_\omega, Z_\omega$ by the following sequence of rotations:

(i) rotation about Z_C by angle Ω producing X'_C, Y'_C, Z'_C or $\mathbf{x}'_C = \mathbf{R}_\Omega \mathbf{x}_C$ where

$$\mathbf{R}_\Omega = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ and X'_C axis points toward ascending node.

(ii) rotation about X'_C by angle ι producing X''_C, Y''_C, Z''_C or $\mathbf{x}''_C = \mathbf{R}_\iota \mathbf{x}'_C$ where

$$\mathbf{R}_\iota = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\iota & \sin\iota \\ 0 & -\sin\iota & \cos\iota \end{bmatrix}$$ and X''_C-Y''_C plane is the orbital plane.

(iii) rotation about Z''_C by angle ω producing X'''_C, Y'''_C, Z'''_C or $\mathbf{x}'''_C = \mathbf{R}_\omega \mathbf{x}''_C$ where

$$\mathbf{R}_\omega = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ and X'''_C points through perigee.

The X'''_C, Y'''_C, Z'''_C system is coincident with the $X_\omega, Y_\omega, Z_\omega$ system and the transformation can be written as $\mathbf{x}_\omega = \mathbf{R}_\omega \mathbf{R}_\iota \mathbf{R}_\Omega \mathbf{x}_C$ or $\mathbf{x}_\omega = \mathbf{R}_{\omega\iota\Omega}\,\mathbf{x}_C$. The rotation matrix $\mathbf{R}_{\omega\iota\Omega}$ is orthogonal, hence its inverse is equal to its transpose, and we may write the transformation $\mathbf{x}_C = \mathbf{R}_{\omega\iota\Omega}^T \mathbf{x}_\omega$ or $\mathbf{x}_C = \mathbf{R}\,\mathbf{x}_\omega$ where $\mathbf{R} = \mathbf{R}_{\omega\iota\Omega}^T$

alternative procedure for steps 6 and 7

In step 7, the transformation from $X_\omega, Y_\omega, Z_\omega = 0$ Perifocal coordinates to X_C, Y_C, Z_C Geocentric–equatorial coordinates can be written as

$$
\begin{aligned}
X_C &= X_\omega \cos\omega\cos\Omega - X_\omega \sin\omega\cos\iota\sin\Omega - Y_\omega \sin\omega\cos\Omega - Y_\omega \cos\omega\cos\iota\sin\Omega \\
Y_C &= X_\omega \cos\omega\sin\Omega + X_\omega \sin\omega\cos\iota\cos\Omega - Y_\omega \sin\omega\sin\Omega + Y_\omega \cos\omega\cos\iota\cos\Omega \\
Z_C &= X_\omega \sin\omega\sin\iota + Y_\omega \cos\omega\sin\iota
\end{aligned}
$$

and substituting $X_\omega = r\cos\theta$ and $Y_\omega = r\sin\theta$ and gathering terms gives

$$
\begin{aligned}
X_C &= \left\{r\left(\cos\theta\cos\omega - \sin\theta\sin\omega\right)\right\}\cos\Omega - \left\{r\left(\cos\theta\sin\omega + \sin\theta\cos\omega\right)\right\}\cos\iota\sin\Omega \\
Y_C &= \left\{r\left(\cos\theta\cos\omega - \sin\theta\sin\omega\right)\right\}\sin\Omega + \left\{r\left(\cos\theta\sin\omega + \sin\theta\cos\omega\right)\right\}\cos\iota\cos\Omega \\
Z_C &= \left\{r\left(\cos\theta\sin\omega + \sin\theta\cos\omega\right)\right\}\sin\iota
\end{aligned}
$$

Using the trigonometric addition formula $\cos(\omega+\theta) = \cos\omega\cos\theta - \sin\omega\sin\theta$ and $\sin(\omega+\theta) = \sin\omega\cos\theta + \cos\omega\sin\theta$ gives

$$
\begin{aligned}
X_C &= r\cos(\omega+\theta)\cos\Omega - r\sin(\omega+\theta)\cos\iota\sin\Omega \\
Y_C &= r\cos(\omega+\theta)\sin\Omega + r\sin(\omega+\theta)\cos\iota\cos\Omega \\
Z_C &= r\sin(\omega+\theta)\sin\iota
\end{aligned}
$$

Introducing an <u>orbit–plane coordinate system</u> $X_P, Y_P, Z_P = 0$ where the X_P-axis points in the direction of the ascending node and the X_P-Y_P plane is the satellite orbital plane with the Y_P-axis advanced 90° in the direction of orbital motion; the angle $u = \omega + \theta$ called <u>the argument of latitude</u> (measured in the plane of the orbit from the line of the nodes) allows the alternative steps, denoted 6* and 7*:

6*. Compute orbit–plane coordinates $X_P, Y_P, Z_P = 0$ from $X_P = r\cos u$, $Y_P = r\sin u$ where the argument of latitude $u = \omega + \theta$

7*. Transform orbit plane coordinates $X_P, Y_P, Z_P = 0$ to Geocentric–equatorial coordinates X_C, Y_C, Z_C using

$$
\begin{aligned}
X_C &= X_P\cos\Omega - Y_P\cos\iota\sin\Omega \\
Y_C &= X_P\sin\Omega + Y_P\cos\iota\cos\Omega \\
Z_C &= Y_P\sin\iota
\end{aligned}
\tag{94}
$$

X_T, Y_T, Z_T TERRESTRIAL COORDINATES AT TIME t_k USING KEPLERIAN ORBITAL ELEMENTS

Using the procedure outlined above, X_C, Y_C, Z_C can be obtained from equation (94). Now consider a rotation from the X_C, Y_C, Z_C reference frame to the X_T, Y_T, Z_T reference frame given by equation (92) which we may write as

$$\begin{aligned} X_T &= X_C \cos\lambda_A - Y_C \cos\lambda_A \\ Y_T &= X_C \sin\lambda_A + Y_C \cos\lambda_A \\ Z_T &= Z_C \end{aligned} \tag{95}$$

Substituting equations (94) into equations (95) gives

$$\begin{aligned} X_T &= \left(X_P \cos\Omega - Y_P \cos\iota \sin\Omega\right)\cos\lambda_A - \left(X_P \sin\Omega + Y_P \cos\iota \cos\Omega\right)\sin\lambda_A \\ Y_T &= \left(X_P \cos\Omega - Y_P \cos\iota \sin\Omega\right)\sin\lambda_A + \left(X_P \sin\Omega + Y_P \cos\iota \cos\Omega\right)\cos\lambda_A \\ Z_T &= Y_P \sin\iota \end{aligned}$$

Expanding and gathering terms gives

$$\begin{aligned} X_T &= X_P\left(\cos\Omega\cos\lambda_A - \sin\Omega\sin\lambda_A\right) - Y_P \cos\iota\left(\sin\Omega\cos\lambda_A + \cos\Omega\sin\lambda_A\right) \\ Y_T &= X_P\left(\cos\Omega\sin\lambda_A - \sin\Omega\cos\lambda_A\right) - Y_P \cos\iota\left(\cos\Omega\cos\lambda_A - \sin\Omega\sin\lambda_A\right) \\ Z_T &= Y_P \sin\iota \end{aligned}$$

Using the trigonometric addition formula $\cos\left(\Omega + \lambda_A\right) = \cos\Omega\cos\lambda_A - \sin\Omega\sin\lambda_A$ and $\sin\left(\Omega + \lambda_A\right) = \sin\Omega\cos\lambda_A + \cos\Omega\sin\lambda_A$ gives the transformation from the <u>orbit–plane coordinate system</u> $X_P, Y_P, Z_P = 0$ to the Conventional Terrestrial coordinates as

$$\begin{aligned} X_T &= X_P \cos\left(\Omega + \lambda_A\right) - Y_P \cos\iota \sin\left(\Omega + \lambda_A\right) \\ Y_T &= X_P \sin\left(\Omega + \lambda_A\right) + Y_P \cos\iota \cos\left(\Omega + \lambda_A\right) \\ Z_T &= Y_P \sin\iota \end{aligned} \tag{96}$$

where $\lambda_A = \left(24^h - \text{GST}\right)\times 15$ is the longitude of the First Point of Aries (in degrees), ι is the inclination of the orbit plane and Ω is the Right Ascension of the ascending node. The orbit-plane coordinates $X_P, Y_P, Z_P = 0$ are computed from step 6* of the procedure for computing X_C, Y_C, Z_C coordinates outline above.

Equations (96) can be further simplified by substituting the longitude of the ascending node $\lambda_N = \lambda_A + \Omega$ from equation (90) to give

$$\begin{aligned} X_T &= X_P \cos\lambda_N - Y_P \cos\iota \sin\lambda_N \\ Y_T &= X_P \sin\lambda_N + Y_P \cos\iota \cos\lambda_N \\ Z_T &= Y_P \sin\iota \end{aligned} \tag{97}$$

APPENDIX: Vectors

Vectors are very useful for describing various physical quantities or relationships such as force, velocity, acceleration, distance between objects, etc., that have both magnitude and direction. Vectors are represented by arrows between points or analytically by symbols such as $\overrightarrow{OP}$, or boldface characters **A** or **a**. The magnitude of a vector is denoted by $\left|\overrightarrow{OP}\right|, |\mathbf{A}|$ or $|\mathbf{a}|$ but it is also common to use A or a to represent the magnitude of vectors **A** or **a**.

A scalar, on the other hand, is a quantity having magnitude but no direction, e.g., mass, length, time, temperature and any real number.

Laws of Vector Algebra: If **A**, **B** and **C** are vectors and m and n are scalars then

1.	$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$	Commutative law for Addition
2.	$\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$	Associative law for Addition
3.	$m\mathbf{A}=\mathbf{A}m$	Commutative law for Multiplication
4.	$m(n\mathbf{A})=(mn)\mathbf{A}$	Associative law for Multiplication
5.	$(m+n)\mathbf{A}=m\mathbf{A}+n\mathbf{A}$	Distributive law
6.	$m(\mathbf{A}+\mathbf{B})=m\mathbf{A}+m\mathbf{B}$	Distributive law

A unit vector is a vector having unit magnitude (a magnitude of one). Unit vectors are denoted by $\hat{\mathbf{A}}$ or $\hat{\mathbf{a}}$ and

$$\hat{\mathbf{A}}=\frac{\mathbf{A}}{|\mathbf{A}|}=\frac{\mathbf{A}}{A}$$

Any vector **A** can be represented by a unit vector $\hat{\mathbf{A}}$ in the direction of **A** multiplied by the magnitude of A. That is, $\mathbf{A}=A\hat{\mathbf{A}}$

In an x,y,z Cartesian reference frame, the vector

$$\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$$

has component vectors $A_1\mathbf{i}$, $A_2\mathbf{j}$ and $A_3\mathbf{k}$ in the x, y and z directions respectively, where **i**, **j** and **k** are unit vectors in the x, y and z directions. A_1, A_2 and A_3 are scalar components. The magnitude of **A** is

$$A=|\mathbf{A}|=\sqrt{A_1^2+A_2^2+A_3^2}$$

The unit vector of **A** is

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A} = \frac{A_1}{A}\mathbf{i} + \frac{A_2}{A}\mathbf{j} + \frac{A_3}{A}\mathbf{k}$$

<u>The scalar product</u> (or dot product) of two vectors $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ is defined as the product of the magnitudes of **A** and **B** multiplied by the cosine of the angle between them, or

$$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta = AB\cos\theta$$

and

$$\mathbf{A} \bullet \mathbf{B} = (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \bullet (B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}) = A_1B_1 + A_2B_2 + A_3B_3$$

Note that $\mathbf{A} \bullet \mathbf{B}$ is a scalar and not a vector.

The following laws are valid for scalar products:

1. $\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$ Commutative law
2. $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$ Distributive law
3. $m(\mathbf{A} \bullet \mathbf{B}) = (m\mathbf{A}) \bullet \mathbf{B} = (\mathbf{A} \bullet \mathbf{B})m$ where m is a scalar
4. $\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1; \quad \mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$
5. $\mathbf{A} \bullet \mathbf{A} = a^2 = a_1^2 + a_2^2 + a_3^2$
6. If $\mathbf{A} \bullet \mathbf{B} = 0$, and **A** and **B** are not null vectors then **A** and **B** are perpendicular.

<u>The vector product</u> (or cross product) of two vectors $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ is a vector $\mathbf{P} = \mathbf{A} \times \mathbf{B}$ where **P** is a vector <u>perpendicular</u> to the plane containing **A** and **B**. The magnitude of **P** is defined as the product of the magnitudes of **A** and **B** multiplied by the sine of the angle between them. The vector product is often expressed as

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\ \hat{\mathbf{P}} = AB\sin\theta\,\hat{\mathbf{P}}$$

where $\hat{\mathbf{P}}$ is a <u>perpendicular unit vector</u> and the direction of **P** is given by the *right-hand-screw rule,* i.e., if **A** and **B** are in the plane of the head of a screw, then a clockwise rotation of **A** to **B** through an angle θ would mean that the direction of **P** would be the same as the direction of advance of a right-handed screw turned clockwise. The cross product can be written as the expansion of a determinant as

$$\mathbf{P} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \overset{(+)}{\mathbf{i}} & \overset{(-)}{\mathbf{j}} & \overset{(+)}{\mathbf{k}} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = (A_2B_3 - A_3B_2)\mathbf{i} - (A_1B_3 - A_3B_1)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}$$

Note here that the mnemonics $(+)$, $(-)$, $(+)$ are an aid to the evaluation of the determinant. The perpendicular vector $\mathbf{P} = P_1\mathbf{i} + P_2\mathbf{j} + P_3\mathbf{k}$ has scalar components $P_1 = (A_2B_3 - A_3B_2)$, $P_2 = -(A_1B_3 - A_3B_1)$ and $P_3 = (A_1B_2 - A_2B_1)$.

The following laws are valid for vector products:

1. $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$ [Commutative law for cross products fails]
2. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ Distributive law
3. $m(\mathbf{A} \times \mathbf{B}) = (m\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (m\mathbf{B}) = (\mathbf{A} \times \mathbf{B})m$ where m is a scalar
4. $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$; $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Triple products

Scalar and vector multiplication of three vectors **A**, **B** and **C** may produce meaningful products of the form $(\mathbf{A} \bullet \mathbf{B})\mathbf{C}$, $\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C})$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. The following laws are valid:

1. $(\mathbf{A} \bullet \mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B} \bullet \mathbf{C})$
2. $\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \bullet (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \bullet (\mathbf{A} \times \mathbf{B})$ (scalar triple products)
3. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$
4. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \bullet \mathbf{C})\mathbf{B} - (\mathbf{A} \bullet \mathbf{B})\mathbf{C}$ (vector triple products)
   $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \bullet \mathbf{C})\mathbf{B} - (\mathbf{B} \bullet \mathbf{C})\mathbf{A}$

Differentiation of vectors

If **A**, **B** and **C** are differentiable vector functions of a scalar u, and φ is a differentiable scalar function of u, then

1. $\dfrac{d}{du}(\mathbf{A} + \mathbf{B}) = \dfrac{d\mathbf{A}}{du} + \dfrac{d\mathbf{B}}{du}$
2. $\dfrac{d}{du}(\mathbf{A} \bullet \mathbf{B}) = \mathbf{A} \bullet \dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du} \bullet \mathbf{B}$
3. $\dfrac{d}{du}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du} \times \mathbf{B}$

4. $\dfrac{d}{du}(\varphi\mathbf{A}) = \varphi\dfrac{d\mathbf{A}}{du} + \dfrac{d\varphi}{du}\mathbf{A}$

5. $\dfrac{d}{du}\{\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C})\} = \mathbf{A}\bullet\left(\mathbf{B}\times\dfrac{d\mathbf{C}}{du}\right) + \mathbf{A}\bullet\left(\dfrac{d\mathbf{B}}{du}\times\mathbf{C}\right) + \dfrac{d\mathbf{A}}{du}\bullet(\mathbf{B}\times\mathbf{C})$

6. $\dfrac{d}{du}\{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\} = \mathbf{A}\times\left(\mathbf{B}\times\dfrac{d\mathbf{C}}{du}\right) + \mathbf{A}\times\left(\dfrac{d\mathbf{B}}{du}\times\mathbf{C}\right) + \dfrac{d\mathbf{A}}{du}\times(\mathbf{B}\times\mathbf{C})$

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